On The Zero-Input Stability of Control Systems With Clegg Integrators

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Abstract

In this paper, we consider control systems utilizing so-called Clegg integrators. A Clegg integrator is a linear integrator with reset mechanism whose describing function $\frac{1.62}{\omega}\angle -38.1°$ has magnitude slope equivalent to that of a linear integrator, but with 51 less degrees of phase lag. The potential advantages of using Clegg integrators to meet stringent design specifications has been demonstrated in the literature, however, stability analyses are missing. This paper addresses the zero-input stability of such control systems and provides a definitive stability condition for second-order plants as well as a providing preliminary results for higher-order plants.

1. Introduction

An integrator is an element found in every control engineer's toolbox helping to provide zero steady-state errors to constant exogenous signals. However, this benefit comes at the expense of 90 degrees of phase lag at all frequencies. The Clegg integrator was developed in [1] to reduce this phase lag while retaining the integrator's desirable magnitude slope frequency response. Indeed, the describing function of a Clegg integrator is $\frac{1.62}{\omega}\angle -38.1°$ with magnitude slope equivalent to that of a linear integrator but with a smaller constant phase response of $-38.1°$. Thus, it seems that the Clegg integrator could potentially provide solutions to control problems involving limited classes of signals; namely sinusoids, with stringent design specifications by overcoming the limitations imposed by Bode's gain-phase relationship; see [2] and [3]. However, except for the sufficient conditions for oscillation and non-oscillation given in the general context of describing functions (e.g., see [5]), the problem of closed-loop stability for the Clegg integrator has not been addressed. The motivation for this paper is to give more definitive stability conditions.

Consider a negative-feedback system consisting of a single-input, single-output, linear time-invariant (LTI) plant with transfer function $G(s) : x_c(s) \to y(s)$ and a *Clegg integrator*: $-y(t) \to x_c(t)$. The action of the Clegg integrator is straightforward: it integrates, except when its input $-y(t)$ crosses zero upon which its output $x_c(t)$ is set to zero. More precisely, let a zero-crossing τ be an instance when $y(\tau) = 0$ and collect all such crossing times in an ordered set $\{t_i : t_i < t_{i+1}, i = 1, 2, \ldots, \}$. Then, the output of a Clegg integrator is described by the dynamical equation:

$$\begin{aligned} x_c(t) &= \int_{t_i}^{t} -y(t)dt, \quad t_i < t < t_{i+1} \\ x_c(t_i) &= 0. \end{aligned} \tag{1}$$

We are interested in the stability of this closed-loop system to initial conditions on both the plant and Clegg integrator and to illustrate, let's consider the simple case when $G(s)$ is a first-order plant

$$G(s) = \frac{k}{s+\alpha} \tag{2}$$

with $k > 0$. Write the state-space equations for the feedback system as

$$\begin{aligned} \dot{x}(t) &= -\alpha x(t) + k x_c(t), \quad t \geq 0; \\ y(t) &= x(t); \\ \dot{x}_c(t) &= -y(t), \quad t_i < t < t_{i+1}; \\ x_c(t_i) &= 0. \end{aligned}$$

For initial conditions $x(0)$ and $x_c(0)$, assume the first zero-crossing of $y(t)$ occurs at t_1. Thus,

$$y(t_1) = x(t_1) = 0; \quad x_c(t_1) = 0$$

which implies

$$x(t) = 0, \;\; x_c(t) = 0$$

for all $t > t_1$. On the other hand, if $y(t) \neq 0$ for all $t \geq 0$, then the Clegg integrator never resets and behaves like a linear integrator. Since this linear closed-loop system is stable,

$$\lim_{t\to\infty} x(t) = 0, \;\; \lim_{t\to\infty} x_c(t) = 0.$$

In either case, the closed-loop system with Clegg integrator is zero-input stable and for first-order plants the Clegg integrator resets at most once and behaves linearly thereafter. However, higher-order plants exhibit more complex behavior, and the remainder of this paper is geared to giving necessary and sufficient conditions for the zero-input stability of Clegg control systems with second-order plants. These results are achieved by showing that the zero-crossings of $y(t)$ are periodically spaced, reducing the stability problem to a linear, shift-invariant discrete-time problem. We illustrate this stability condition (see Theorem 1) with a simple example.

2. Second-Order Plants

Assume that the plant $G(s) = c(sI - A)^{-1}b$ is a second-order transfer function. Using (1), this closed-loop system has a state-space representation:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bx_c(t); \\ y(t) &= cx(t); \\ \dot{x}_c(t) &= -y(t), \quad t_i < t < t_{i+1}; \\ x_c(t_i) &= 0 \end{aligned} \tag{3}$$

where $A \in \mathbf{R}^{2\times 2}$, $b \in \mathbf{R}^{2\times 1}$, $c \in \mathbf{R}^{1\times 2}$, $x(t)$ is the plant state and $x_c(t)$ is the Clegg integrator's state. The zero-crossing times of $y(t)$ are collected in the ordered set $\{t_i : t_i \le t_{i+1},\ i = 1, 2, \ldots, \}$. Between successive crossing-times the plant state behaves as the LTI system

$$x(t) = P(t - t_i)x(t_i), \quad t \in (t_i, t_{i+1}) \tag{4}$$

where

$$P(t) = [I \quad 0]\, e^{\bar{A}t} \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{5}$$

and

$$\bar{A} = \begin{bmatrix} A & b \\ -c & 0 \end{bmatrix}.$$

By definition, a *crossing time* t_i is characterized by $cx(t_i) = 0$. Thus, from (4), the crossing times t_i must satisfy:

$$\begin{aligned} cx(t_1) &= 0 \\ cP(\tau_1)x(t_1) &= 0 \\ cP(\tau_2)P(\tau_1)x(t_1) &= 0 \\ &\vdots \end{aligned} \tag{6}$$

where $\tau_i = t_{i+1} - t_i$. We say that the zero-crossings t_i are *periodically-spaced* whenever (6) is satisfied with $\tau_1 = \tau_2 = \ldots = $. The next lemma further characterizes zero-crossings for second-order plants.

Lemma 1: *The closed-loop system consisting of plant $G(s)$ and Clegg integrator as described in (3) has the following properties:*

1. *If c is an not an eigenvector of $P(\tau)$ for all $\tau > 0$, then $y(t)$ has one zero-crossing t_1 at most for which $x(t_1) = 0$.*
2. *If c is an eigenvector of $P(\tau)$ for some $\tau > 0$, then the zero-crossings are infinite in number and periodically-spaced. Moreover, the smallest such τ is the spacing of the zero-crossings.*

Proof: *1.* Suppose c is an not an eigenvector of $P(\tau)$ for all $\tau > 0$. Then,

$$det \begin{bmatrix} c \\ cP(\tau) \end{bmatrix} \neq 0 \tag{7}$$

for all $\tau > 0$. Proceeding by contradiction, suppose there are two zero-crossings t_1 and t_2. Then, $cx(t_1) = 0$ and $cP(t_2 - t_1)x(t_1) = 0$ which together with both (7) and $P(t) \in \mathbf{R}^{2\times 2}$ implies $x(t_1) = 0$. In addition, $x_c(t_1) = 0$ so that $x(t) = 0$ and $x_c(t) = 0$ for all $t > t_1$ contradicting the existence of t_2.

2. Suppose there exists at least one zero-crossing t_1; i.e., $cx(t_1) = 0$. Let $\tau > 0$ be the smallest number for which $cP(\tau) = \lambda c$ for some λ. Hence, τ is the smallest τ_1 for which $cP(\tau_1)x(t_1) = 0$ and

$$det \begin{bmatrix} c \\ cP(\tau_1) \end{bmatrix} = 0.$$

Now, because $P(t) \in \mathbf{R}^{2\times 2}$, τ is also the smallest τ_2 such that

$$det \begin{bmatrix} c \\ cP(\tau) \\ cP(\tau_2)P(\tau) \end{bmatrix} = 0.$$

Thus, τ is the smallest τ_2 for which $cP(\tau_2)P(\tau)x(t_1) = 0$. Continuing this argument shows $\tau = \tau_1 = \tau_2 = \ldots =$ so that the zero-crossings are indeed periodic. Finally, since $x(t_1) \neq 0$ and $P(\tau)$ is nonsingular, then

$$x(t_2) = P(\tau)x(t_1) \neq 0.$$

Similarly, $x(t_i) \neq 0$ for $i = 3, 4, \ldots, \infty$. Thus, the zero-crossings are infinite in number. □

Remark: From Lemma 1, $y(t)$ has either an infinite number of periodically-spaced zero-crossings or at most, one crossing. In the latter case, (3) is zero-input stable if and only if $G(s)$ is stabilized by a linear integrator. In the former case, when there is an infinite number of periodic zero-crossings, stability may be deduced from the state's behavior at these periodic instances. Our main result exploits this periodicity and gives a stability condition in terms of a shift-invariant, discrete-time system.

Theorem 1: *The closed-loop system consisting of plant $G(s)$ and Clegg integrator as described in (3) is zero-input stable if and only if:*

1. *$G(s)$ is stabilized by a linear integrator when c is not an eigenvector of $P(\tau)$ for all $\tau > 0$.*
2. *All the eigenvalues of $P(\tau)$ are contained in the open unit disk when $\tau > 0$ is the smallest number for which c is an eigenvector of $P(\tau)$.*

Proof: In the first case, it follows from Lemma 1 that $y(t)$ has one zero crossing at most. After this crossing, the Clegg integrator behaves as a linear integrator which is assumed to stabilize $G(s)$. In the second case, it follows from Lemma 1 that the zero-crossings are infinite in number and periodic. Hence, from (4), the plant state satisfies the state equation

$$x(t_{i+1}) = P(\tau)x(t_i), \quad i = 1, 2, \ldots, \infty$$

which is asymptotically stable since the eigenvalues of $P(\tau)$ are contained in the open unit disk. This prove the theorem.

$\square$

3. Example

Consider the feedback system where

$$G(s) = \frac{(3+\beta)s + 1}{s^2 + 3s - \beta}$$

and β is a free parameter. For all β, $G(s)$ is stabilized by a linear integrator. Similarly, for all β, $\tau = 2$ is the smallest positive number for which c is an eigenvector of $P(\tau)$. Thus, from Lemma 1, the Clegg integrator resets every 2 seconds, independent of β. The largest eigenvalue of $P(2)$ is $(5 + 2\beta)e^{-2}$; hence, if we choose β such that

$$\left|(5 + 2\beta)e^{-2}\right| < 1,$$

then the eigenvalues of $P(\tau)$ will be contained in the open unit disk. From Theorem 1, the system is zero-input stable. On the other hand, if

$$\left|(5 + 2\beta)e^{-2}\right| \geq 1,$$

then the system is zero-input unstable. Hence, this Clegg control system is zero-input stable if and only if $\beta \in (-6.195, 1.195)$. It is interesting to see that describing function analysis inaccurately "predicts" the stability range to be $(-4.9, 1.75)$. Finally, while this example shows that a plant stabilized by a linear integrator may or may not be stabilized by a Clegg integrator, an application of Theorem 1 shows that

$$G(s) = \frac{(3.1)s + 0.3322}{s^2 + 3s - 0.1}$$

is stabilized by the Clegg integrator even though it is not stabilized by the linear integrator.

4. Higher-Order Plants

In contrast to the results in Lemma 1, higher-order control systems can exhibit complex reset patterns. In this section we give some preliminary results for higher-order plants and state a sufficient condition for stability. To begin, again consider the set-up in (3) where now $A \in \mathbf{R}^{n\times n}$, $b \in \mathbf{R}^{n\times 1}$ and $c \in \mathbf{R}^{1\times n}$. With $P(t)$ as in (4) and (5), we define

$$\mathcal{T}(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) = \begin{bmatrix} c \\ cP(\lambda_1) \\ cP(\lambda_2)P(\lambda_1) \\ \vdots \\ cP(\lambda_{n-1})\ldots P(\lambda_1)x(t_1) \end{bmatrix}.$$

Theorem 2: *Let the nth-order plant $G(s) = c(sI - A)^{-1}b$ be stabilizable by an integrator. If $\mathcal{T}(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ is nonsingular for all $\lambda_i > 0$, $i = 1, 2, \ldots, n-1$, then the Clegg integrator resets $n-1$ times at most and the feedback system described in (3) is zero-input stable.*

Proof: We first show that $y(t)$ has $n-1$ zero-crossings at most. Indeed, suppose there are more. Then, the first n equations of (6) are satisfied for some λ_i, $i = 1, 2, \ldots, n-1$. Since $\mathcal{T}(\lambda_1, \lambda_2, \ldots, \lambda_{n_1})$ is nonsingular, $x(t_1) = 0$. This implies $x(t) = 0$ and $x_c(t) = 0$ for all $t > t_1$ which contradicts the existence of t_2. Therefore, the largest number of zero crossings is $n-1$ and the Clegg eventually behaves as a linear integrator which is assumed to stabilize $G(s)$.

$\square$

Remark: The zero-crossing behavior described in Theorem 2 is akin to the first case of Lemma 1.

5. Conclusion

This paper completely resolves the question of zero-input stability for a negative-feedback system involving a Clegg integrator and a second-order plant transfer function. First, we show that the Clegg integrator resets periodically and then prove that stability can be deduced from an underlying linear, shift-invariant discrete-time system. For higher-order plants we give a sufficient condition for stability.

References

[1] J. C. Clegg, "A Nonlinear Integrator for Servomechanism," *Trans A.I.E.E.* Part II, 77, pp. 41–42, 1958.

[2] K. R. Krishnan and I. M. Horowitz, "Synthesis of a Nonlinear Feedback System with Significant Plant-Ignorance for Prescribed System Tolerance," *International Journal of Control*, Vol. 19, pp. 689–706, 1974.

[3] I. M. Horowitz and P. Rosenbaum, "Nonlinear Design for Cost of Feedback Reduction in Systems with Large Parameter Uncertainty," *International Journal of Control*, Vol. 21, pp. 977–1001, 1975.

[4] C.T. Chen, *Linear System Theory and Design*, Holt, Rinehart and Winston, 1984.

[5] H.K. Khalil, *Nonlinear Systems*, Macmillan, 1992.