Fixed Point of a TCP/RENO DiffServ Network with a Single Congested Link

Y. Chait and C.V. Hollot
Technical Note DACS02-07
Department of Mechanical and Industrial Engineering
University of Massachusetts Amherst

8.20.03

Abstract

This technical note includes lengthy technical details referred to by our various DiffServ papers.

1 Preliminaries

The DiffServ network under consideration run TCP RENO, uses a multi-PI AQM for marking at the core and token bucket for coloring at the edges (see later papers in this URL for fluid flow modelling details). The network relations at equilibrium are given by:

$$\begin{aligned} \frac{2(1-p_i)}{p_i} &= W_i^2 \\ p_i &= (1-f_{gi})p_r + f_{gi}p_g \end{aligned}$$

where W_i denotes the window size of a generic flow within the source aggregate, p_i is the source's apparent marking probability, and p_g and p_r are the green and red packet marking probabilities at the differentiating core. The source aggregate rate is

$$x_i = \alpha_i^{-1} W_i$$

where

$$\alpha_i = \frac{\tau_i}{\eta_i}$$

where τ_i denotes the sources round trip time and η_i is the number of flows comprising the source (i.e., load factor). In a token bucket setup, the fraction of packets colored green is computed from (A denotes bucket rate and r denotes aggregate send rate)

$$f_{gi} = \min\left\{1, \frac{A_i}{x_i}\right\} \in [0, 1].$$

From the window equation, the source rate is related to the marking probability as follows

$$x_i = \alpha_i^{-1}\sqrt{\frac{2}{p_i} - 2}.$$

Before we proceed with the fixed point problem, we describe a limitation of the current leaky bucket implementation. The present implementation of leaky buckets (see f_{gi} relation above) often forces the equilibrium queue level to be at the green set-point even at an exact-provisioned network. Since the green set-point is above the red set-point, so is the queuing delay. The reason for this is as follows. In an exact-provisioned case, Theorem 2 below shows that all sources exactly meet their target $\underline{x}_i$. In certain network settings the necessary token bucket rate of source i, A_i, is such that $A_i \geq x_i$ giving $f_{gi} = 1$. Hence, the lack of red packets in this source implies that the queue must settle at the green set-point.

We formalize the above discussion in the context of both PI-ARM and multi-level PI-AQM controllers. The result is presented without proof.

Lemma 1: *If a PI-ARM is used, then*

$$\begin{cases} x_i < \underline{x}_i & \Rightarrow & A_i \to \infty \ \ (f_{gi} = 1) \\ x_i > \underline{x}_i & \Rightarrow & A_i = 0 \ \ (f_{gi} = 0) \end{cases}.$$

Moreover, Let q_{red} and q_{green} denote the set-points for the multi-level PI-AQM. We have

$$\begin{cases} p_r < 1 & \Rightarrow & p_g = 0 \ \ and \ \ q_0 = q_{red} \\ p_r = 1 & \Rightarrow & \begin{cases} p_g > 0 \ \ and \ \ q_0 = q_{green}, \ \ or \\ p_g = 0 \ \ and \ \ q_{red} \leq q_0 < q_{green} \end{cases} \end{cases}.$$

□

1.1 Exact-Provisioned Case

Let the number of sources with target rates be n. The core capacity is denoted by C.

Theorem 2: *If*

$$\sum_{i=1}^{n} \underline{x}_i = C$$

then

$$x_i = \underline{x}_i, \ \ i = 1, \ldots, n.$$

Proof: Proceeding by contradiction, suppose that $x_{i^*} > \underline{x}_{i^*}$ for some $i^* \in \{1, 2, \ldots, n\}$. Then from Lemma 1 $A_{i^*} = 0$, hence $f_{ri^*} = 1$. Thus, $p_r < 1$ which in turn implies, using Lemma 1, that $p_g = 0$. Also $\sum_{i=1,\ldots,n} \underline{x}_i = C$ implies that $x_{\hat{i}} < \underline{x}_{\hat{i}}$ for some $\hat{i} \neq i^*$. But, from

Lemma 1, $x_{\hat{i}} - \underline{x}_{\hat{i}} < 0$ produces $A_{\hat{i}} = \infty$, or $f_{g\hat{i}} = 1$. Since $p_g = 0$, then $x_{\hat{i}} \to \infty$, which is a contradiction. Hence, no source rate lags or exceeds its target rate. □

The equilibrium point may not be unique and the following scenarios are possible. If

$$1 \geq p_r > \max_{i \in [1,\ldots,n]} \frac{2\alpha_i^{-2}}{\underline{x}_i^2 - 2\alpha_i^{-2}}$$

then due to the optimization nature of TCP RENO (reported in several articles), the queue will settle at the red set-point q_{red} which is the minimal cost. However, if $p_r = 1$ is not sufficient to regulate rates to their targets, the link will need to increase cost. This can come in the form of increased queuing delay. In this instance, $q_{red} < q < q_{green}$ is feasible if

$$1 > \max_{i \in [1,\ldots,n]} \frac{2\alpha_i^{-2}}{\underline{x}_i^2 - 2\alpha_i^{-2}}$$

where the increased queuing delay is included in τ_i. If this is not feasible, the queue will settle at its highest possible level which is the green set-point q_{green}. Again, due to the implicit optimization, it follows that

$$p_g = \max_{i \in [1,\ldots,n]} \frac{2\alpha_i^2}{\underline{x}_i^2 - 2\alpha_i^{-2}}.$$

□

1.2 Over-Provisioned Case

Without loss of generality we assume that the sources are ordered such that

$$\alpha_1 \underline{x}_1 \geq \alpha_2 \underline{x}_2 \geq \ldots \geq \alpha_n \underline{x}_n. \tag{1}$$

Before we present our main result, we need the following intermediate result.

Lemma 3: *If*

$$\sum_{i=1}^{n} \underline{x}_1 < C \tag{2}$$

then:

(i) The queue's state at equilibrium is $q_o = q_{red}$ and $p_r < 1$.

(ii) All sources achieve their target rates at the least:

$$x_i \geq \underline{x}_i, \quad i = 1, \ldots, n.$$

(iii) If in addition to (2), $x_{i^*} > \underline{x}_{i^*}$ for some $i^* \in \{1, \ldots, n-1\}$, then

$$x_i > \underline{x}_i, \quad \forall i \in \{i^*, \ldots, n\}.$$

(iv) If, in addition to (2), $x_{i^*} = \underline{x}_{i^*}$ and $x_{i^*+1} > \underline{x}_{i^*+1}$ for some $i^* \in \{1, \ldots, n-1\}$, then

$$\alpha_i^* \underline{x}_i^* \geq \alpha_j x_j, \quad j \in \{i^* = 1, \ldots n\}.$$

Proof: To prove (i), due to excess capacity, at least one source exceeds its target rate, say $x_{i^*} > \underline{x}_{i^*}$ for some $i^* \in \{1, \ldots, n\}$. From Lemma 1, $A_{i^*} = 0$ so $f_{ri^*} = 1$. Since $x_{i^*} = \alpha_{i^*}^{-1}\sqrt{\frac{2}{p_r} - 2}$, and, $x_{i^*} > \underline{x}_{i^*} > 0$, then $0 < p_r < 1$. From Lemma 1 $q_o = q_{red}$ which proves (i).

To show (ii) we proceed by contradiction. Assume that $x_{i^*} < \underline{x}_{i^*}$ for some $i^* \in \{1, \ldots, n\}$. From Lemma 1 it follows that $A_{i^*} \to \infty$ implying $f_{gi^*} = 1$. From (i) and Lemma 1, $p_g = 0$ so it follows that $x_{i^*} \to \infty$. A contradiction.

To show (iii) we first use the fact that $x_{i^*} > \underline{x}_{i^*}$ implies $f_{ri^*} = 1$. So

$$x_{i^*} = \alpha_{i^*}^{-1}\sqrt{\frac{2}{p_r} - 2}. \tag{3}$$

From (i) $p_g = 0$, thus, for any rate x_i:

$$x_i = \alpha_i^{-1}\sqrt{\frac{2}{f_{ri}p_r} - 2} \geq \alpha_i^{-1}\sqrt{\frac{2}{p_r} - 2}.$$

Specifically, for $i > i^*$, combining the above with (1) and (3) gives

$$x_i \geq \alpha_i^{-1}\frac{x_{i^*}}{\alpha_{i^*}^{-1}} > \alpha_i^{-1}\frac{\underline{x}_{i^*}}{\alpha_{i^*}^{-1}} \geq \alpha_i^{-1}\frac{\underline{x}_i}{\alpha_i^{-1}} = \underline{x}_i.$$

This proves (iii).

Finally, to prove (iv), start with $x_{i^*} = \underline{x}_{i^*}$. Then

$$\alpha_{i^*}\underline{x}_{i^*} = \alpha_{i^*}\underline{x}_{i^*} = \sqrt{\frac{2}{f_{ri^*}p_r} - 2} \geq \sqrt{\frac{2}{p_r} - 2}.$$

Now $x_j > \underline{x}_j$ implies that

$$\alpha_j x_j = \sqrt{\frac{2}{p_r} - 2}$$

Combining the above gives

$$\alpha_{i^*}\underline{x}_{i^*} \geq \alpha_{i^*+1}x_{i^*+1}.$$

$\square$

The following is our main result for an over-provisioned network.

Theorem 4: *If*

$$\sum_{i=1}^{n} x_i < C$$

and i^* denotes the smallest integer in $\{1, 2, \ldots, n\}$ such that

$$\frac{C - \sum_{i=1}^{i^*-1} \underline{x}_i}{\sum_{i=i^*}^{n} \alpha_i^{-1}} - \alpha_i^* \underline{x}_i^* > 0, \tag{4}$$

then,

$$x_i = \begin{cases} = \underline{x}_i, & i = 1, \ldots, i^* - 1 \\ > \underline{x}_i, & i = i^*, \ldots, n. \end{cases} \tag{5}$$

Moreover, the greedy flows $\{x_i : \ i = i^*, \ldots, n\}$ grab the available capacity according to:

$$x_i = \alpha_i^{-1} \frac{C - \sum_{j=1}^{i^*-1} \underline{x}_j}{\sum_{j=i^*}^{n} \alpha_j^{-1}}.$$

Proof: First, we show that there exists an i^* satisfying (4). Since $C > \sum_{i=1}^{n} \underline{x}_i$, then it follows that

$$\alpha_n \left(C - \sum_{i=1}^{n-1} \underline{x}_i \right) - \alpha_n \underline{x}_n > 0.$$

Hence, $i^* = n$ always satisfies (4).

Next, we prove (17). From Lemma 3(ii) there exists an $\hat{\imath} \in \{1, 2, \ldots n\}$ such that

$$x_i = \begin{cases} = \underline{x}_i, & i = 1, \ldots, \hat{\imath} - 1 \\ > \underline{x}_i, & i = \hat{\imath}, \ldots, n. \end{cases} \tag{6}$$

Proceeding by contradiction, assume $\hat{\imath} \neq i^*$. We consider two cases.

(*Case 1:* $\hat{\imath} < i^*$) In this situation

$$C = \sum_{i=1}^{\hat{\imath}-1} \underline{x}_i + \sum_{i=\hat{\imath}}^{n} x_i. \tag{7}$$

Recall from Lemma 3(iii) that

$$\alpha_{\hat{\imath}} x_{\hat{\imath}} = \alpha_{\hat{\imath}+1} x_{\hat{\imath}+1} = \ldots = \alpha_n x_n.$$

Combining this with (7) gives

$$\frac{C - \sum_{i=1}^{\hat{\imath}-1} \underline{x}_i}{\sum_{i=\hat{\imath}}^{n} \alpha_i^{-1}} = \alpha_{\hat{\imath}} x_{\hat{\imath}}.$$

By assumption $x_{\hat{\imath}} > \underline{x}_{\hat{\imath}}$, so

$$\frac{C - \sum_{i=1}^{\hat{\imath}-1} \underline{x}_i}{\sum_{i=\hat{\imath}}^{n} \alpha_i^{-1}} - \alpha_{\hat{\imath}} \underline{x}_{\hat{\imath}} > 0.$$

Using (4) this implies $i^* \leq \hat{i}$, which is a contradiction.

(*Case 2:* $\hat{i} > i^*$) It follows from (6) that $x_{i^*} = \underline{x}_{i^*}$. Thus, from Lemma 3(iv),

$$\alpha_{i^*}\underline{x}_{i^*} \geq \alpha_i x_i$$

for $i = \hat{i}, \ldots, n$ so that

$$\begin{aligned} C &= \sum_{i=1}^{\hat{i}-1} \underline{x}_i + \sum_{i=\hat{i}}^{n} \alpha_i^{-1} \frac{x_i}{\alpha_i^{-1}} \\ &= \sum_{i=1}^{i^*-1} \underline{x}_i + \sum_{i=i^*}^{\hat{i}-1} \underline{x}_i + \sum_{i=\hat{i}}^{n} \alpha_i^{-1} \frac{x_i}{\alpha_i^{-1}} \\ &\leq \sum_{i=1}^{i^*-1} \underline{x}_i + \sum_{i=i^*}^{\hat{i}-1} \alpha_i^{-1} \frac{\underline{x}_i}{\alpha_i^{-1}} + \sum_{i=\hat{i}}^{n} \alpha_i^{-1} \frac{\underline{x}_{i^*}}{\alpha_{i^*}^{-1}} \\ &\leq \sum_{i=1}^{i^*-1} \underline{x}_i + \sum_{i=i^*}^{\hat{i}-1} \alpha_i^{-1} \frac{\underline{x}_{i^*}}{\alpha_{i^*}^{-1}} + \sum_{i=\hat{i}}^{n} \alpha_i^{-1} \frac{\underline{x}_{i^*}}{\alpha_{i^*}^{-1}} \\ &\leq \sum_{i=1}^{i^*-1} \underline{x}_i + \alpha_{i^*}\underline{x}_{i^*} \sum_{i=i^*}^{n} \alpha_i^{-1} \end{aligned}$$

which contradicts (4). This completes the proof. □

The marking/loss probability can now be computed. Since for i^* in (4)

$$\alpha_{i^*} x_{i^*} = \sqrt{\frac{2}{p_r} - 2}$$

and

$$x_i = \alpha_i^{-1} \frac{C - \sum_{j=1,\ldots,k-1} \underline{x}_j}{\sum_{j=k,\ldots,n} \alpha_j^{-1}}$$

we obtain

$$p_r = \frac{1}{1 + 0.5\left(\frac{C - \sum_{i=1,\ldots,i^*-1} \underline{x}_i}{\sum_{i=i^*,\ldots,n} \alpha_i^{-1}}\right)^2}.$$

1.2.1 Interpretation of Theorem 4

The index i^* defines which sources are the most "greedy" in terms of grabbing available bandwidth. For each source, the product $\alpha_i \underline{x}_i$ defines the required window size to achieve its target rate $\underline{x}_i$. And this window defines the required marking/loss probability $p_i = f_{ri} p_r$.

While the multi-level AQM computes p_r necessary for congestion control, the ARM attempts to adjust f_{ri} such that p_i remains fixed in spite of network variations. Hence, the role of the ARM is to override TCP's attempts to have all sources experience the same marking/loss probability. As long as ARM finds an $f_{ri} \geq 1$, we achieve $x_i = \underline{x}_i$. When there is excess network capacity, p_r may become sufficiently low such that even with $f_{ri} = 1$, $x_i > \underline{x}_i$. Either way, with p_i fixed for each source, those sources whose ARM must resort to setting $f_{ri} = 1$ are the "greedy" ones since they are solely under the control of TCP and hence have same window length while sharing the excess bandwidth.

1.3 Over-Provisioned Case with External Sources

Consider a DiffServ network with sources that do not have a target rate. Specifically, let the $\{1, \ldots, n\}$ sources have target rates $\underline{x}_i$ while the $n+1$ source does not, hence all its packets are marked red.

Theorem 5: *If*

$$\sum_{i=1}^{n} \underline{x}_i < C \tag{8}$$

then Lemma 3 hold for all the subscribing flows i=1,...,n. In addition, if i^* is an integer such that

$$i^* = \begin{cases} \min_{i \in [1,\ldots,n]} \frac{C - \sum_{j=1}^{i-1} \underline{x}_j}{\sum_{j=i}^{n+1} \alpha_j^{-1}} - \alpha_i \underline{x}_i > 0, & \frac{C - \sum_{j=1}^{n-1} \underline{x}_j}{\sum_{j=n}^{n+1} \alpha_j^{-1}} - \alpha_n \underline{x}_n > 0 \\ n+1, & \textit{otherwise.} \end{cases} \tag{9}$$

then

$$x_i = \begin{cases} = \underline{x}_i, & i = 1, \ldots, i^* - 1 \\ > \underline{x}_i, & i = i^*, \ldots, n. \end{cases} \tag{10}$$

Moreover, the greedy flows $\{x_i : \ i = i^*, \ldots, n\}$ grab the available capacity according to:

$$x_i = \alpha_i^{-1} \frac{C - \sum_{j=1}^{i^*-1} \underline{x}_j}{\sum_{j=i^*}^{n+1} \alpha_j^{-1}} \tag{11}$$

while the external source shares the bandwidth according to

$$x_{n+1} = \alpha_{n+1}^{-1} \frac{C - \sum_{j=1}^{i^*-1} \underline{x}_j}{\sum_{j=i^*}^{n+1} \alpha_j^{-1}} \tag{12}$$

Proof: We begin by proving Lemma 3 for this setup. To show 3(i), note that $p_{n+1} = p_r$. If $q_0 > q_{red}$, then $p_r = 1$ and $x_{n+1} = 0$ resulting in $\sum_{i=1}^{n} \underline{x}_i = C$. So at least one source exceeds its target rate, say $x_{i^*} > \underline{x}_{i^*}$ for some $i^* \in \{1, \ldots, n\}$. From Lemma 1, $A_{i^*} = 0$ so

$f_{ri^*} = 1$. Since $x_{i^*} = \alpha_{i^*}^{-1}\sqrt{\frac{2}{p_r} - 2}$ and $p_r = 1$ implying $x_{i^*} = 0$. A contradiction. Hence at equilibrium $q_o = q_{red}$.

To show 3(ii) we proceed by contradiction. Assume that $x_{i^*} < \underline{x}_{i^*}$ for some $i^* \in \{1, \ldots, n\}$. From Lemma 1 it follows that $A_{i^*} \to \infty$ implying $f_{gi^*} = 1$. From 3(i) and Lemma 1, $p_g = 0$ so it follows that $x_{i^*} \to \infty$. A contradiction.

To show 3(iii) we first use the fact that $x_{i^*} > \underline{x}_{i^*}$ for some $i^* \in \{1, \ldots, n\}$ implies $f_{ri^*} = 1$. So

$$x_{i^*} = \alpha_{i^*}^{-1}\sqrt{\frac{2}{p_r} - 2}. \tag{13}$$

From 3(i) $p_g = 0$, thus, for any rate x_i in $i \in \{1, \ldots, n\}$:

$$x_i = \alpha_i^{-1}\sqrt{\frac{2}{f_{ri}p_r} - 2} \geq \alpha_i^{-1}\sqrt{\frac{2}{p_r} - 2}.$$

Specifically, for $i > i^*$, the above, (1) and (13) together give

$$x_i \geq \alpha_i^{-1}\frac{x_{i^*}}{\alpha_{i^*}^{-1}} > \alpha_i^{-1}\frac{\underline{x}_{i^*}}{\alpha_{i^*}^{-1}} \geq \alpha_i^{-1}\frac{\underline{x}_i}{\alpha_i^{-1}} = \underline{x}_i.$$

This proves 3(iii).

Finally, to prove 3(iv), start with $x_{i^*} = \underline{x}_{i^*}$ for some $i^* \in \{1, \ldots, n\}$. Then

$$\alpha_{i^*}\underline{x}_{i^*} = \alpha_{i^*}x_{i^*} = \sqrt{\frac{2}{f_{ri^*}p_r} - 2} \geq \sqrt{\frac{2}{p_r} - 2}.$$

Now $x_{i^*} > \underline{x}_{i^*}$ implies that

$$\alpha_{i^*+1}x_{i^*+1} = \sqrt{\frac{2}{p_r} - 2}$$

Combining the above gives

$$\alpha_{i^*}\underline{x}_{i^*} \geq \alpha_{i^*+1}x_{i^*+1}.$$

To show (10)-(12), we first note that from 3(ii) we know that all sources achieve at least their target rates and that the "ordering" rules 3(iii)-3(iv) also hold in this case. What we have to show is how much bandwidth, if any, x_{n+1} is allocated by TCP. The subscribing sources will then have $C - x_{n+1} \geq \sum_{i=1}^{n} \underline{x}_i$ bandwidth left which is precisely the case solved in Theorem 5. From (9) we have two possibilities: $i \in \{1, \ldots, n\}$ or $i = n+1$. Let $i^* = n+1$ and proceed by contradicting (10) and assuming $x_n > \underline{x}_n$ and $x_{n-1} = \underline{x}_{n-1}$. Using 3(iii)-3(iv) we expand (8) as

$$C = \sum_{i=1}^{n-1} \underline{x}_i + x_n + x_{n+1}.$$

Since the non-subscribing source x_{n+1} has only red packets, as well as any source exceeding their target rates, their rates are given by the generic relation

$$x = \alpha^{-1}\sqrt{\frac{2}{p_r} - 2}$$

where $\sqrt{\frac{2}{p_r}-2}$ is the window W afforded to each such source by TCP. Plugging this into the above expansion gives

$$C-\sum_{i=1}^{n-1}\underline{x}_i=(\alpha_n^{-1}+\alpha_{n+1}^{-1})\sqrt{\frac{2}{p_r}-2}$$

so

$$\frac{C-\sum_{i=1}^{n-1}\underline{x}_i}{\sum_{i=n}^{n+1}\alpha_i^{-1}}=\sqrt{\frac{2}{p_r}-2}=\alpha_n x_n>\alpha_n\underline{x}_n.$$

This contradicts the definition of $i^*=n+1$ in (9), hence we must have $x_n=\underline{x}_n$. From Lemma 3 it follows that $x_i=\underline{x}_i, i=1,\ldots,n$ while the excess bandwidth is grabbed by the non-subscribing source

$$x_{n+1}=C-\sum_{i=1}^{n}\underline{x}_i.$$

Next, consider the other possibility in (9) where $i^*\in[1,\ldots,n]$. Let $i^*=n$ and proceed by contradicting (10) and assuming $x_n=\underline{x}_n$. Since

$$\underline{x}_n=\sqrt{\frac{2}{f_r^n p_r}-2}\geq\sqrt{\frac{2}{p_r}-2}$$

then

$$\begin{aligned}C-\sum_{i=1}^{n-1}\underline{x}_i &= \underline{x}_n+x_{n+1}\\ &= \alpha_n^{-1}\frac{\underline{x}_n}{\alpha_n^{-1}}+\alpha_{n+1}^{-1}\sqrt{\frac{2}{p_r}-2}\\ &\leq \alpha_n^{-1}\frac{\underline{x}_n}{\alpha_n^{-1}}+\alpha_{n+1}^{-1}\sqrt{\frac{2}{f_r^n p_r}-2}\\ &= (\alpha_n^{-1}+\alpha_{n+1}^{-1})\sqrt{\frac{2}{f_r^n p_r}-2}\\ &= (\alpha_n^{-1}+\alpha_{n+1}^{-1})\frac{\underline{x}_n}{\alpha_n^{-1}}\end{aligned}$$

which can be written as

$$\frac{C-\sum_{i=1}^{n-1}\underline{x}_i}{\sum_{i=n}^{n+1}\alpha_i^{-1}}-\alpha_n\underline{x}_n\leq 0.$$

This contradicts the definition of $i^*=n$ in (9), hence we must have $x_n>\underline{x}_n$. To complete proving (10), we continue by contradiction and assume that $x_{n-1} > \underline{x}_{n-1}$. Hence,

$$\begin{aligned} C - \sum_{i=1}^{n-2} \underline{x}_i &= x_{n-1} + x_n + x_{n+1} \\ &= \sqrt{\frac{2}{p_r} - 2} \sum_{i=n-1}^{n+1} \alpha_i^{-1} \\ &= \alpha_{n-1} x_{n-1} \sum_{i=n-1}^{n+1} \alpha_i^{-1} \\ &> \alpha_{n-1} \underline{x}_{n-1} \sum_{i=n-1}^{n+1} \alpha_i^{-1} \end{aligned}$$

which implies $i^* = n-1$ in (9). A contradiction. This shows complete the proof of (10). Finally, using (10) in expanding (8) it is straightforward to show (11)-(12). This completes the proof. □

Remark: In the above case, there's no guarantee that $i^* \in \{1, \ldots, n\}$. For some network parameters, it may be that the $n+1$ source grabs all the available capacity $C - \sum_{i=1}^{n} \underline{x}_i$. This depends on the size of α_{n+1}^{-1}, specifically, how it reduces the available window test relation in (9).

1.4 Under-Provisioned Case

Lemma 6: *If*

$$\sum_{i=1}^{n} \underline{x}_i > C \tag{14}$$

then:

(i) The queue's state at equilibrium is $q_o = q_{green}$.

(ii) All sources at most achieve their target rates:

$$x_i \le \underline{x}_i, \quad i = 1, \ldots, n.$$

(iii) If in addition to (14), $x_{i^*} < \underline{x}_{i^*}$ *for some* $i^* \in \{1, \ldots, n\}$, *then*

$$x_i < \underline{x}_i, \quad \forall i \in \{1, \ldots, i^{\}}].$$

(iv) If, in addition to (14), $x_{i^*} = \underline{x}_{i^*}$ *for some* $i^* \in \{1, \ldots, n\}$ *and* $x_{i*-1} < \underline{x}_{i^*-1}$, *then*

$$\alpha_{i^*} \underline{x}_{i^*} \le \alpha_j x_j, \quad j \in \{1, \ldots, i^* - 1\}.$$

Proof: To prove (i), due to lack of capacity, at least one source does not achieve its target rate, say $x_{i^*} < \underline{x}_{i^*}$ for some $i^* \in \{1, \ldots, n\}$. From Lemma 1, $A_{i^*} \to \infty$ so $f_{ri^*} = 0$. Since $x_{i^*} = \alpha_{i^*}^{-1}\sqrt{\frac{2}{p_g} - 2}$, then $0 < p_g < 1$. From Lemma 1 $q_o = q_{green}$ which proves (i).

To show (ii) we proceed by contradiction. Assume that $x_{i^*} > \underline{x}_{i^*}$ for some $i^* \in \{1, \ldots, n\}$. From Lemma 1 it follows that $A_{i^*} = 0$ implying $f_{ri^*} = 1$. From (i) and Lemma 1, $p_r = 1$ so it follows that $x_{i^*} = \alpha_{i^*}^{-1}\sqrt{\frac{2}{p_r} - 2} = 0$. A contradiction.

To show (iii) we first use the fact that $x_{i^*} < \underline{x}_{i^*}$ implies $f_{gi^*} = 1$. So

$$x_{i^*} = \alpha_{i^*}^{-1}\sqrt{\frac{2}{p_g} - 2}. \tag{15}$$

From (i) $p_r = 1$, thus, for any rate x_i:

$$x_i = \alpha_i^{-1}\sqrt{\frac{2}{f_{ri}(1-p_g) + p_g} - 2} \leq \alpha_i^{-1}\sqrt{\frac{2}{p_g} - 2}.$$

Specifically, for $i < i^*$, the above, (1) and (15) together give

$$x_i \leq \alpha_i^{-1}\frac{x_{i^*}}{\alpha_{i^*}^{-1}} < \alpha_i^{-1}\frac{\underline{x}_{i^*}}{\alpha_{i^*}^{-1}} \leq \alpha_i^{-1}\frac{\underline{x}_i}{\alpha_i^{-1}} = \underline{x}_i.$$

This proves (iii).

Finally, to prove (iv), start with $x_{i^*} = \underline{x}_{i^*}$. Then

$$\alpha_{i^*}\underline{x}_{i^*} = \alpha_{i^*}x_{i^*} = \sqrt{\frac{2}{f_{ri^*}(1-p_g) + p_g} - 2} \leq \sqrt{\frac{2}{p_g} - 2}.$$

Now $x_j < \underline{x}_j$ implies that

$$\alpha_j x_j = \sqrt{\frac{2}{p_g} - 2}$$

Combining the above gives

$$\alpha_{i^*}\underline{x}_{i^*} \leq \alpha_{i^*-1}x_{i^*-1}.$$

□

The following is our main result for an under-provisioned network.

Theorem 7: *If*

$$\sum_{i=1}^{n} \underline{x}_i > C$$

and i^* *denotes the largest integer in* $\{1, 2, \ldots, n\}$ *such that*

$$\alpha_i^* \underline{x}_i^* - \frac{C - \sum_{i=i^*+1}^{n} \underline{x}_i}{\sum_{i=1}^{i^*} \alpha_i^{-1}} > 0, \tag{16}$$

then,

$$x_i = \begin{cases} < \underline{x}_i, & i = 1 \ldots, i^* \\ = \underline{x}_i, & i = i^* + 1, \ldots, n. \end{cases} \tag{17}$$

Moreover, the non-greedy sources $\{x_i : \ i = 1, \ldots, i^*\}$ *under-achieve according to:*

$$x_i = \alpha_i^{-1} \frac{C - \sum_{i=i^*+1}^{n} \underline{x}_i}{\sum_{i=1}^{i^*} \alpha_i{-1}}.$$

Proof: First, we show that there exists an i^* satisfying (16). We consider two possible cases. In the first case, no source achieves its target rate. From Lemma 1, all under-achieving sources are described by

$$x_i = \alpha_i^{-1} \sqrt{\frac{2}{p_g} - 2}$$

so

$$C = \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \alpha_i^{-1} \sqrt{\frac{2}{p_g} - 2}.$$

Rearranging gives

$$\frac{C}{\sum_{i=1}^{n} \alpha_i^{-1}} = \sqrt{\frac{2}{p_g} - 2} = \alpha_i x_i < \alpha_i \underline{x}_i.$$

Hence $i^* = n$ satisfies (16). The second case assumes some sources achieve their target rates. Lemma 6(iii) shows that in such a case, at the least, $x_n = \underline{x}_n$. Assume that $x_i = \underline{x}_i$ only for the n'th source, hence, the remaining sources under-achieve. That is

$$C = \sum_{i=1}^{n-1} x_i + \underline{x}_n = \sum_{i=1}^{n-1} \alpha_i^{-1} \sqrt{\frac{2}{p_g} - 2} + \underline{x}_n$$

which can written as

$$\frac{C - \underline{x}_n}{\sum_{i=1}^{n-1} \alpha_i^{-1}} = \sqrt{\frac{2}{p_g} - 2} = \alpha_{n-1} x_{n-1} < \alpha_{n-1} \underline{x}_{n-1}.$$

Hence $i^* = n - 1$ satisfies (16). This proves that there exists $i^* \in \{1, \ldots, n\}$ satisfying (16).

Next, we prove (17). From Lemma 6(ii) there exists an $\hat{i} \in \{1, 2, \ldots n\}$ such that

$$x_i = \begin{cases} < \underline{x}_i, & i = 1 \ldots, \hat{i} \\ = \underline{x}_i, & i = \hat{i} + 1, \ldots, n. \end{cases} \tag{18}$$

Proceeding by contradiction, assume $\hat{i} \neq i^*$. We consider two cases.

(*Case 1:* $\hat{i} > i^*$) In this situation

$$C = \sum_{i=1}^{\hat{i}} x_i + \sum_{i=\hat{i}+1}^{n} \underline{x}_i. \tag{19}$$

Recall from Lemma 6(iii) that

$$\alpha_{\hat{i}} x_{\hat{i}} = \alpha_{\hat{i}-1} x_{\hat{i}-1} = \ldots = \alpha_1 x_1.$$

Combining this with (19) gives

$$\frac{C - \sum_{i=\hat{i}+1}^{n} \underline{x}_i}{\sum_{i=1}^{\hat{i}} \alpha_i^{-1}} = \alpha_{\hat{i}} x_{\hat{i}}.$$

By assumption, $x_{\hat{i}} < \underline{x}_{\hat{i}}$, so

$$\alpha_{\hat{i}} \underline{x}_{\hat{i}} - \frac{C - \sum_{i=\hat{i}+1}^{n} \underline{x}_i}{\sum_{i=1}^{\hat{i}} \alpha_i^{-1}} > 0.$$

From (16) this implies $\hat{i} \leq i^*$, which is a contradiction.

(*Case 2:* $\hat{i} < i^*$) It follows from (6) that $x_{i^*} = \underline{x}_{i^*}$. Thus, from Lemma 6(iv),

$$\alpha_{i^*} \underline{x}_{i^*} \leq \alpha_i x_i,$$

for $i = 1, \ldots, \hat{i} - 1$, so that

$$\begin{aligned} C &= \sum_{i=1}^{\hat{i}} x_i + \sum_{i=\hat{i}+1}^{n} \underline{x}_i \\ &= \sum_{i=1}^{\hat{i}} x_i + \sum_{\hat{i}+1}^{i^*} x_i + \sum_{i=i^*+1}^{n} \underline{x}_i \\ &\leq \sum_{i=1}^{\hat{i}} x_i + \sum_{i=\hat{i}+1}^{n} \underline{x}_i \\ &= \alpha_{\hat{i}} x_{\hat{i}} \sum_{i=1}^{\hat{i}} \alpha_i^{-1} + \sum_{i=\hat{i}+1}^{n} \underline{x}_i. \end{aligned}$$

This contradicts (16), since given i^*, then the inequality (16) also holds for $i \in \{1, \ldots, i^*\}$. This completes the proof. $\square$