Submitted to IEEE Transactions on Automatic Control

Stability and Asymptotic Performance Analysis of a Class of Reset Control Systems*

Qian Chen† Yossi Chait‡ and C.V. Hollot§

7 August 2000

Abstract

Bode's gain-phase relationship places a hard limitation on performance tradeoffs in linear, time-invariant feedback control systems. It has long been suggested that reset control has the potential to improve this situation. Recent experimental studies support this claim. This paper focuses on the analysis of such reset control systems which has been missing in this past work. Specifically, we give results on bounded-input bounded-output stability, asymptotic stability and steady-state performance. These results are applied to an experimental demonstration of reset control of a flexible mechanism.

*This material is based upon work supported by the National Science Foundation under Grant No.CMS-9800612.

†MIE Department, University of Massachusetts, Amherst, MA 01003; chen@ecs.umass.edu.

‡MIE Department, University of Massachusetts, Amherst, MA 01003; chait@ecs.umass.edu.

§ECE Department, University of Massachusetts, Amherst, MA 01003; hollot@ecs.umass.edu.

1 Introduction

It is well-appreciated that Bode's gain-phase relationship [1] places a hard limitation on performance tradeoffs in linear, time-invariant (LTI), feedback control systems. Specifically, the need to minimize the open-loop high-frequency gain often competes with required high levels of low-frequency loop gains and phase margin bounds. Our focus on reset control systems is motivated by its potential to improve this situation as demonstrated thoretically in [2][1] and by simulations and experiments [3]-[6].

The basic concept in reset control is to reset the state of a linear controller to zero whenever its input meets a threshold. Typical reset controllers include the so-called Clegg integrator [7] and first-order reset element (FORE) [3]. The former is a linear integrator whose output resets to zero when its input crosses zero. The latter generalizes the Clegg concept to a first-order lag filter. In [7], the Clegg integrator was shown to have a describing function similar to the frequency response of a linear integrator but with only 38.1° phase lag. A FORE was shown to have similar feature while providing a further design freedom when compared with Clegg-integrator ([4] and [6]). In our study, we adopt the FORE reset mechanism in feedback interconnection with a linear system to obtain the so-called *reset control system* shown in Figure 1. The signals r, y, e, n and d in Figure 1 represent reference input, output, error signal, sensor noise and disturbance, respectively, and $L(s)$ denotes the linear loop consisting of the plant and any linear compensation[2].

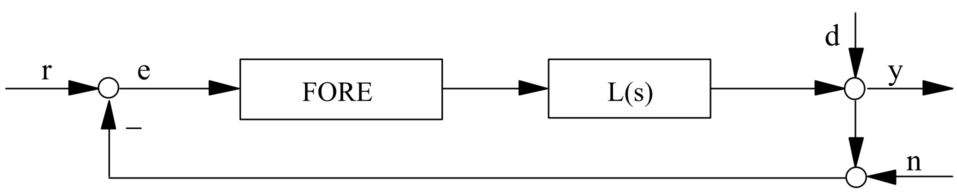

Figure 1: The class of reset control systems considered in this paper.

The objective of this paper is to provide a level of analysis missing in past work on reset control. The analysis in [7] was limited to describing functions while [3] and [4] ignored stability issues altogether. An application of small gain in [5] appears too conservative and could not validate the observed experimental performance in [6]. Motivated by this lack of results, this paper continues our recent work reported in a sequence of conference papers [8], [9], and [10]. In this paper, we introduce a condition, called the *β positive-real condition*, which, when satisfied, allows one to assert BIBO and asymptotic stability of the reset control system. Under this condition, we will also show that the reset control system inherits the steady-state tracking properties of an underlying linear control system. Very importantly,

[1]This work provides an example of control specifications that can be achieved by reset control and not by linear feedback.

[2]The design of the reset control system in Figure 1 involves the selection of *both* the FORE's pole and some linear compensation in $L(s)$. This will be discussed in Section 5.

we will show that the β positive-real condition is satisfied for the experiment considered in [6], thus confirming the observed stability as well as demonstrating the applicability of our results.

Reset control action resembles a number of popular nonlinear control strategies including relay control [11], sliding mode control [12] and switching control [13]. A common feature to these is the use of a switching surface to trigger change in control signal. Distinctively, reset control employs the same (linear) control law on both sides of the switching surface. Resetting occurs when the system trajectory impacts this surface. This reset action can be alternatively viewed as the injection of judiciously-timed, state-dependent impulses into an otherwise LTI feedback system. This analogy is evident in the paper where we use impulsive differential equations; e.g., see [14] and [15], to model dynamics. Despite this relationship, we found existing theory on impulse differential equations to be either too general or broad to be of immediate and direct use. Finally, this connection to impulsive control helps to draw comparison to a body of control work [16] where impulses were introduced in an open-loop fashion to quash oscillations in vibratory systems.

The paper is organized as follows. In Section 2 we set-up a model to describe the reset control system in Figure 1 and identify a key underlying linear control system which we refer to as the *base-linear system*. Section 3 is central. It introduces this notion of β positive-realness and links it to BIBO stability. In Section 4 we again use the β positive-real condition to show that the base-linear system passes-on its steady-state performance properties to the reset control system. In Section 5, we apply these results to an experimental system involving speed control of a flexible mechanism which will demonstrate the non-trivial applicability of our analytical results. It also provides another independent and favorable comparison of reset to LTI control.

2 Set-Up

In this paper we focus on the reset control system in Figure 1 where the first-order reset element (FORE) is described by the impulsive differential equation [14]:

$$\begin{array}{rcll} \dot{x}_f(t) & = & -bx_f(t) + e(t); & e(t) \neq 0 \\ x_f(t^+) & = & 0; & e(t) = 0 \end{array}$$

where x_f is its state, e is the system error and b the FORE's pole; see [3]. To avoid degeneration to a LTI system, we assume that the FORE continually resets. We collect these reset times in the unbounded set

$$I = \{t_i \mid e(t_i) = 0,\ t_i > t_{i-1} + \sigma,\ \sigma > 0,\ i = 1, 2, ..., \infty\}$$

where we assume that adjacent reset times are no closer than σ. This assumption is technically motivated by a desire to have closed-loop solutions continuable over $[0, \infty)$, but is met when FORE is digitally implemented and the sampling period is a lower bound to σ.

A state-space description of the reset control system is:

$$\begin{aligned} \dot{x}_p(t) &= Ax_p(t) + Bx_f(t) \\ \dot{x}_f(t) &= -Cx_p(t) - bx_f(t) + w(t); \quad t \notin I \\ x_f(t^+) &= 0; \qquad\qquad\qquad\qquad\quad t \in I \\ y(t) &= Cx_p(t) + d(t) \end{aligned} \tag{1}$$

where $\{A, B, C\}$ denotes a minimal realization of $L(s)$, $x_p(t) \in \Re^n$ and $w(t) \triangleq r(t) - n(t) - d(t)$ is the aggregate input signal. Given $(x_p(0), x_f(0))$, the solution to (1) is piecewise left-continuous on the intervals $(t_i, t_{i+1}]$. In the absence of resetting, (1) reduces to the following linear system:

$$\begin{bmatrix} \dot{x}_{pl}(t) \\ \dot{x}_{fl}(t) \end{bmatrix} \triangleq A_{cl} \begin{bmatrix} x_{pl}(t) \\ x_{fl}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ w(t) \end{bmatrix}; \quad \begin{bmatrix} x_{pl}(0) \\ x_{fl}(0) \end{bmatrix} = \begin{bmatrix} x_p(0) \\ x_f(0) \end{bmatrix} \tag{2}$$

where

$$A_{cl} = \begin{bmatrix} A & -B \\ -C & -b \end{bmatrix}.$$

We refer to this as the *base-linear system* and, in the sequel, we will show that it can pass on some of its properties, such as stability and asymptotic performance, to its associated reset control system.

3 BIBO Stability Analysis

In this section we analyze the BIBO stability of (1) which requires every bounded input[3] w to produce a bounded output y. To begin this analysis we apply the transformation

$$\begin{aligned} z_p(t) &\doteq x_p(t) - x_{pl}(t) \\ z_f(t) &\doteq x_f(t) - x_{fl}(t) \end{aligned} \tag{3}$$

to (1) to obtain:

$$\begin{aligned} \dot{z}_p(t) &= Az_p(t) + Bz_f(t) \\ \dot{z}_f(t) &= -Cz_p(t) - bz_f(t); \quad t \notin I \\ z_f(t_i^+) &= -x_{fl}(t_i); \qquad\qquad t \in I. \end{aligned} \tag{4}$$

As an intermediate step, we show that boundedness of z_p implies that y is bounded.

Lemma 1: *Assume A_{cl} is asymptotically stable and r, d and n are bounded. If z_p is bounded, then output y is bounded.*

[3] A signal z is said to *bounded* if there exists a constant M such that $|z(t)| < M$ for all t.

Proof: We have

$$\begin{aligned} |y(t)| &= |Cx_p(t) + d(t)| \\ &\leq |Cz_p(t)| + |Cx_{pl}(t)| + |d(t)|. \end{aligned}$$

Since A_{cl} is stable and w is bounded, then x_{pl} is bounded. Output y is thus bounded. $\square$

Before we present our main result on BIBO stability, we need the following lemmas.

Lemma 2: *If A_{cl} is asymptotically stable and w is bounded, there exists constants M_1 and M_2 such that $|z_f(t_i^+)| < M_1$ and $|Cz_p(t_i)| < M_2$ for $i = 1, 2, \ldots, \infty$.*

Proof: Because A_{cl} is asymptotically stable and w, then x_{fl} and x_{pl} are bounded. From (4), $z_f(t_i^+) = -x_{fl}(t_i)$. Therefore, there exists an M_1 such that $|z_f(t_i^+)| < M_1$ for $i = 1, 2, \ldots, \infty$. By definition, $Cz_p(t_i) = w(t_i) - Cx_{pl}(t_i)$. Since w and x_{pl} are bounded, then there exists an M_2 such that $|Cz_p(t_i)| < M_2$ for $i = 1, 2, \ldots, \infty$. $\square$

The next is the well-known Meyer-Kalman-Yakubovich Lemma [17].

Lemma 3: *Let $Z(s) = h(sI - F)^{-1}g$ be a scalar transfer function where H is asymptotically stable. If $Z(s)$ is strictly positive-real*[4] *then there exist a symmetric positive-definite matrix P, a vector q, and a positive constant ε such that*

$$\begin{aligned} F^T P + PF &= -q^T q - \varepsilon P; \\ Pg &= h^T. \end{aligned}$$

Our next definition introduces a positive-real condition that is key in establishing the results of this paper.

Definition 1: The reset control system (1) is said to satisfy the β *positive-real condition* if there exists a $\beta \in \Re$ such that

$$h(s) \stackrel{\triangle}{=} [\beta C \quad 1](sI - A_{cl})^{-1}[0 \ \cdots \ 0 \ 1]^T \quad \textit{is strictly positive-real.} \tag{5}$$

We now state a main result:

Theorem 1: *The reset control system* (1) *is BIBO stable if the β positive-real condition* (5) *is satisfied.*

Proof: Since $h(s)$ in (5) is strictly positive-real, then, from Lemma 3, there exists a positive-definite matrix P, a vector q and a positive constant ε such that

$$\begin{aligned} PA_{cl} + A_{cl}^T P &= -q^T q - \varepsilon P; \\ P[0 \ \cdots \ 0 \ 1]^T &= [\beta C \quad 1]^T. \end{aligned} \tag{6}$$

[4] A transfer function $X(s)$ is said to be *strictly positive real* if: (i) $X(s)$ is asymptotically stable, and (ii) $\mathrm{Re}[X(j\omega)] > 0, \ \forall \omega \geq 0$.

Hence, P can be written as

$$P = \begin{bmatrix} P_1 & \beta C^T \\ \beta C & 1 \end{bmatrix}$$

where $P_1 \in \Re^{n\times n}$ is positive-definite. Along the piecewise left-continuous solutions of (4) we define

$$\begin{aligned} V(t) &= [z_p^T(t), z_f(t)]P[z_p^T(t), z_f(t)]^T \\ &= z_p^T(t)P_1 z_p(t) + 2\beta C z_p(t) z_f(t) + z_f^2(t) \end{aligned}$$

over $t \in (t_i, t_{i+1}]$. At the reset instants $t = t_i$ we then have

$$\begin{aligned} V(t_i^+) &= z_p^T(t_i)P_1 z_p(t_i) + 2\beta C z_p(t_i) z_f(t_i^+) + z_f^2(t_i^+) \\ &= V(t_i) + 2\beta C z_p(t_i) z_f(t_i^+) + z_f^2(t_i^+) - 2\beta C z_p(t_i) z_f(t_i) - z_f^2(t_i). \end{aligned}$$

Since $-2\beta C z_p(t_i) z_f(t_i) - z_f^2(t_i) \le (\beta C z_p(t_i))^2$,

$$\begin{aligned} V(t_i^+) &\le V(t_i) + 2\beta C z_p(t_i) z_f(t_i^+) + z_f^2(t_i^+) + (\beta C z_p(t_i))^2 \\ &= V(t_i) + [z_f(t_i^+) + \beta C z_p(t_i)]^2. \end{aligned} \tag{7}$$

Because w is bounded, it follows from Lemma 2 that there exists a constant $M > 0$ such that $[z_f(t_i^+) + \beta C z_p(t_i)]^2 \le M$ for $i = 1, 2, \ldots, \infty$. Thus, from (7):

$$V(t_i^+) \le V(t_i) + M, \quad i = 1, 2, \ldots, \infty.$$

Differentiating $V(t)$ along solutions to (4), we use (6) to obtain

$$\begin{aligned} \dot{V}(t) &= [z_p^T(t), z_f(t)](PA_{cl} + A_{cl}^T P)[z_p^T(t), z_f(t)]^T \\ &= [z_p^T(t), z_f(t)](-q^T q - \varepsilon P)[z_p^T(t), z_f(t)]^T \\ &\le -\varepsilon[z_p^T(t), z_f(t)]P[z_p^T(t), z_f(t)]^T \\ &= -\varepsilon V(t) \end{aligned}$$

for all $t \in (t_i, t_{i+1}]$. The non-negativity of $V(t)$ implies

$$V(t) \le e^{-\varepsilon(t-t_i)} V(t_i^+) \tag{8}$$

whenever $t \in (t_i, t_{i+1}]$. Since $t_{i+1} - t_i > \sigma$,

$$\begin{aligned} V(t_{i+1}) &\le e^{-\varepsilon(t_{i+1}-t_i)} V(t_i^+) \\ &\le e^{-\varepsilon\sigma} V(t_i^+) \\ &\le e^{-\varepsilon\sigma}[V(t_i) + M]. \end{aligned}$$

Combining this with (8) gives

$$V(t) \le e^{-\varepsilon(t-t_i)}[e^{-\varepsilon(i-1)\sigma} V(0) + M + e^{-\varepsilon\sigma} M + \ldots + e^{-\varepsilon(i-1)\sigma} M] \tag{9}$$

for all $t \in (t_i, t_{i+1}]$. Since $V(0) = 0$, $V(t) \leq M/(1-e^{-\varepsilon\sigma})$. Therefore, V is bounded. Because P is positive-definite, it follows that z_p is bounded. Finally, from Lemma 1, y is bounded. This completes the proof. □

Remarks: (i) While the β positive-real condition is only sufficient for BIBO stability, it appears that it may be applicable to non-trivial situations. For example, in Section 5 we show that this condition is satisfied for a reset control system having 12^{th}-order $L(s)$. Similarly, in [18], the experimental set-up in [6] is shown to satisfy the β positive-real condition (5).

(ii) There exists an important class of reset control systems that satisfy the β positive-real condition and, hence, are BIBO stable. To describe them, consider the reset control systems in Figure 1 with

$$L(s) = \frac{(s+b)\omega_n^2}{s(s+2\zeta\omega_n)}$$

where b is the pole of FORE and $\zeta, \omega_n > 0$. This class was introduced in [3] and its base-linear system has standard second-order transfer function

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

This class of reset control systems satisfies the β positive-real condition (5) for all combination of positive parameters b, ζ and ω_n; see [9]. Therefore, from Theorem 1, these reset control systems are BIBO stable.

(iii) It is possible that a reset control system is unstable even though its base-linear system is stable and describing-function analysis does not predict a limit-cycle. Such an example is given in [8].

3.1 Robustness to Implementation Errors

In (1) we implicitly assumed that the reset process is ideal; that is, the state of FORE resets exactly to zero at the precise instant when its input $e(t)$ is zero. Of course, this seldom happens as exemplified by the digital implementation of reset elements where such errors occur due to finite sampling rates and signal quantization. To account for such inaccuracies, we modify the model of reset control accordingly to:

$$\begin{aligned}
\dot{x}_p(t) &= Ax_p(t) + Bx_f(t) \\
\dot{x}_f(t) &= -Cx_p(t) - bx_f(t) + w(t); \quad t \notin I \\
x_f(t^+) &= \epsilon_1(t); \qquad\qquad\qquad\qquad t \in I \\
I &= \{t : Cx_p(t) = w(t) + \epsilon_2(t),\ t_{i+1} - t_i > \sigma, \sigma > 0, i = 1, 2, ...\},
\end{aligned} \tag{10}$$

where ϵ_1 and ϵ_2 are bounded signals modeling implementation errors. The boundedness of ϵ_2 is necessary for y to be bounded. The following corollary states that the BIBO stability condition in Theorem 1 remains valid even in the face of these implementation errors.

Corollary 1: *The reset control system* (10) *is BIBO stable if it satisfies the* β *positive-real condition* (5).

Proof: The proof follows along the same lines as that in Theorem 1. After using the state transformation (3), system (10) becomes:

$$\begin{aligned}
\dot{z}_p(t) &= Az_p(t) + Bz_f(t) \\
\dot{z}_f(t) &= -Cz_p(t) - bz_f(t); \quad t \notin I \\
z_f(t_i^+) &= -x_{fl}(t_i) + \epsilon_1(t); \quad\quad t \in I.
\end{aligned}$$

Since ϵ_1 is bounded, it is straightforward to show that Lemma 1 and Lemma 2 are still in effect. Taking the same V and following through the proof of Theorem 1 yields bounded V, z_p, and, finally, bounded y. This completes the proof. □

4 Asymptotic Analysis

In this section we show that satisfaction of the β positive-real condition (5) yields more than BIBO stability. With it, we can further show that the reset control system (1) is *asymptotically stable* and that it inherits the *asymptotic tracking* properties of its base-linear system. In the sequel we denote the tracking error in the reset control system and its base-linear system by

$$e(t) = w(t) - Cx_p(t); \quad e_l(t) = w(t) - Cx_{pl}(t),$$

respectively. We first need the following technical lemmas.

Lemma 4: *If* $\lim_{t\to\infty} e_l(t) = 0$, *then*

$$\lim_{t\to\infty} Cz_p(t_i) = 0 \text{ and } \lim_{i\to\infty} z_f(t_i^+) = 0.$$

Proof: From the definition of t_i,

$$Cz_p(t_i) = w(t_i) - Cx_{pl}(t_i) \to 0,$$

as $i \to \infty$. From (2) we have

$$\dot{x}_{fl}(t) = -bx_{fl}(t) - Cx_{pl}(t) + w(t).$$

Since $\lim_{t\to\infty} e_l(t) = 0$ and $b > 0$, then $\lim_{t\to\infty} x_{fl}(t) = 0$. From (4), $z_f(t_i^+) = -x_{fl}(t_i)$ so that $\lim_{i\to\infty} z_f(t_i^+) = 0$. □

Lemma 5: *If the* β *positive-real condition* (5) *is satisfied and* $\lim_{t\to\infty} e_l(t) = 0$, *then*

$$\lim_{t\to\infty} \left(\begin{bmatrix} x_p(t) \\ x_f(t) \end{bmatrix} - \begin{bmatrix} x_{pl}(t) \\ x_{fl}(t) \end{bmatrix} \right) = 0.$$

Proof: Take $V(t)$ as in the proof of Theorem 1. Then, from (7),

$$V(t_i^+) \leq V(t_i) + [z_f(t_i^+) + \beta C z_p(t_i)]^2.$$

With $M_i = [z_f(t_i^+) + \beta C z_p(t_i)]^2$ and $\lim_{t\to\infty} e_l(t) = 0$, it follows from Lemma 4 that $\lim_{i\to\infty} M_i = 0$. Thus, (9) becomes

$$V(t) \leq e^{-\varepsilon(t-t_i)}[e^{-\varepsilon(i-1)\sigma}V(0) + M_i + e^{-\varepsilon\sigma}M_{i-1} + \ldots + e^{-\varepsilon(i-1)\sigma}M_1] \tag{11}$$

for all $t \in (t_i, t_{i+1}]$. Since $V(0) = 0$, then from (4) $\lim_{t\to\infty} V(t) = 0$ so that

$$\lim_{t\to\infty}\left(\begin{bmatrix} x_p(t) \\ x_f(t) \end{bmatrix} - \begin{bmatrix} x_{pl}(t) \\ x_{fl}(t) \end{bmatrix}\right) = 0.$$

This completes the proof. $\square$

We know state our asymptotic stability result.

Theorem 2: *The reset control system* (1) *is asymptotically stable if it satisfies the* β *positive-real condition* (5).

Proof: Set $w(t) \equiv 0$. From (11), it is straightforward to compute

$$V(t) \leq \sup_i \frac{M_i}{1 - e^{-\varepsilon\sigma}}$$

where $M_i = [z_f(t_i^+) + \beta C z_p(t_i)]^2$. From (4), $z_f(t_i^+) = -x_{fl}(t_i)$ and from (3), $C z_p(t_i) = -C x_{pl}(t_i)$ so that $M_i = [-x_{fl}(t_i) - \beta C x_{pl}(t_i)]^2$. Therefore,

$$V(t) \leq \sup_{t_i}[-x_{fl}(t_i) - \beta C x_{pl}(t_i)]^2/(1 - e^{-\varepsilon\sigma}). \tag{12}$$

The right-hand side of (12) can be bounded as in

$$[x_{fl}(t) + \beta C x_{pl}(t)]^2 \leq k \left\| \begin{matrix} x_{pl}(t) \\ x_{fl}(t) \end{matrix} \right\|^2$$

for some $k > 0$ and for all $t > 0$. Since $V(t)$ is a positive-definite function, then the left-hand side of (12) can be bounded below by the norm of $[z_p^T(t), z_f(t)]^T$. Hence, there exists a constant k_1 such that

$$\left\| \begin{matrix} z_p(t) \\ z_f(t) \end{matrix} \right\| \leq k_1 \sup_{t\in[0,\infty)} \left\| \begin{matrix} x_{pl}(t) \\ x_{fl}(t) \end{matrix} \right\|$$

for all $t > 0$. Therefore, from (3),

$$\left\| \begin{matrix} x_p(t) \\ x_f(t) \end{matrix} \right\| \leq (k_1 + 1) \sup_{t\in[0,\infty)} \left\| \begin{matrix} x_{pl}(t) \\ x_{fl}(t) \end{matrix} \right\|. \tag{13}$$

Since the base-linear system (2) is asymptotically stable and

$$x_{pl}(0) = x_p(0); \quad x_{fl}(0) = x_f(0),$$

then (13) implies that (1) is Lyapunov stable. To complete the proof we need to show that the state asymptotically converges. Since A_{cl} is stable then, from (2),

$$\lim_{t\to\infty} \begin{bmatrix} x_{pl}(t) \\ x_{fl}(t) \end{bmatrix} = 0.$$

Therefore, from Lemma 5,

$$\lim_{t\to\infty} \begin{bmatrix} x_p(t) \\ x_f(t) \end{bmatrix} = 0$$

showing that the states asymptotically converge. This proves the theorem. □

We now show that the base-linear system can pass on its asymptotic tracking properties to its reset control system.

Theorem 3: *Suppose the β positive-real condition* (5) *is satisfied. If* $\lim_{t\to\infty} e_l(t) = 0$, *then* $\lim_{t\to\infty} e(t) = 0$.

Proof: From Lemma 5, $\lim_{t\to\infty}[Cx_p(t) - Cx_{pl}(t)] = 0$. Consequently, $\lim_{t\to\infty} e(t) = 0$. This completes the proof. □

Theorem 3 indicates that the classical "type k" behavior of a base-linear system is inherited by its reset control system. Specifically, if $r(t)$ and $d(t)$ are polynomials signals of degree no greater than k, if $lim_{t\to\infty} n(t) = 0$ and if $L(s)$ contains at least k integrators, then the reset system (1) has zero steady-state tracking error provided it satisfies the β positive-real condition (5).

5 Speed Control of a Flexible Mechanical System

In this section we apply reset control design to the speed control of the rotational flexible mechanical system shown in Figure 2. This system consists of three inertias connected via flexible shafts. A servo motor drives inertia J_3 and the speed of inertia J_1 is measured via a tachometer. The controller is implemented using dSPACE tools [19]. A more complete description of this experiment can be found in [18]. Besides introducing readers to some of the details behind reset control design, one objective of this section is to demonstrate the applicability of the main theoretical results of this paper; namely, Theorems 1-3. First, we consider LTI feedback control.

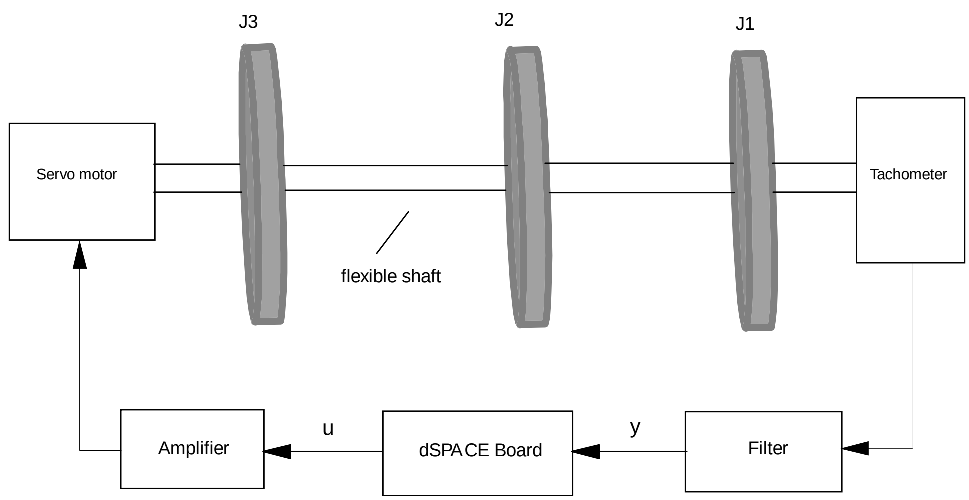

Figure 2: Schematic of the rotational flexible mechanical system.

5.1 Tradeoffs in LTI feedback control

A block diagram of a linear feedback control system is shown in Figure 3 where $C(s)$ denotes the controller and $P(s)$ is a transfer function model of the open-loop system from input u to output y. Using experimental frequency-response data, we identified the plant as

$$P(s) = \frac{46083950}{(s + 1.524)(s^2 + 3.1s + 2820)(s^2 + 3.62s + 9846)}.$$

We pose the following specifications on a stabilizing LTI controller $C(s)$ to illustrate the limitations and tradeoffs in LTI design.

1. *Bandwidth constraint*: The unity-gain cross-over frequency ω_c, defined by $|L(j\omega_c)| = 1$, must satisfy $\omega_c > 3\pi$.

2. *Disturbance rejection*: Low-frequency disturbances are to be rejected; specifically,

$$\left|\frac{y(j\omega)}{d(j\omega)}\right| \leq 0.2, \quad \text{when} \quad \omega \leq \pi;$$

3. *Sensor-noise suppression*: High-frequency sensor noise is to be suppressed; i.e.,

$$\left|\frac{y(j\omega)}{n(j\omega)}\right| \leq 0.3, \quad \text{when} \quad \omega \geq 10\pi;$$

4. *Asymptotic performance*: Zero steady-state tracking error to constant reference r and disturbance d signals.

5. *Overshoot*: Overshoot in output y to a constant reference r should be less than 20%.

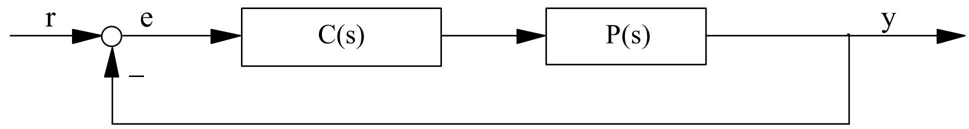

Figure 3: Block diagram of the linear feedback system.

In terms of Bode specifications, the first two constraints translate into minimum-gain requirements on the open-loop gain $|L(j\omega)|$ at low frequencies while the third specification places an upper bound on this gain at high frequencies. The fourth specification requires $C(s)$ to contain an integrator and the fifth specification requires a phase margin of approximately 45°; assuming second-order dominance.

Using classical loop-shaping techniques we were unable to meet all of the above specifications. To illustrate the tradeoffs, consider two candidate, stabilizing LTI controllers:

$$C_1(s) = \frac{1281489(s+4.483)(s^2+3.735s+2851)(s2+5.158s+10060)}{s(s^2+295.1s+22330)(s^2+126.2s+8889)(s^2+239s+27560)}$$

and

$$C_2(s) = \frac{1075460(s+7)(s^2+3.662s+2798)(s^2+5.419s+9876)}{s(s+209.6)(s+35.8)(s^2+132.8s+12050)(s^2+375.9s+66930)}.$$

Figure 4 compares the Bode plots of the corresponding loops $L_1(j\omega) = C_1(j\omega)P(j\omega)$ and $L_2(j\omega) = C_2(j\omega)P(j\omega)$. Loop L_1 fails to satisfy the sensor-noise suppression specification at

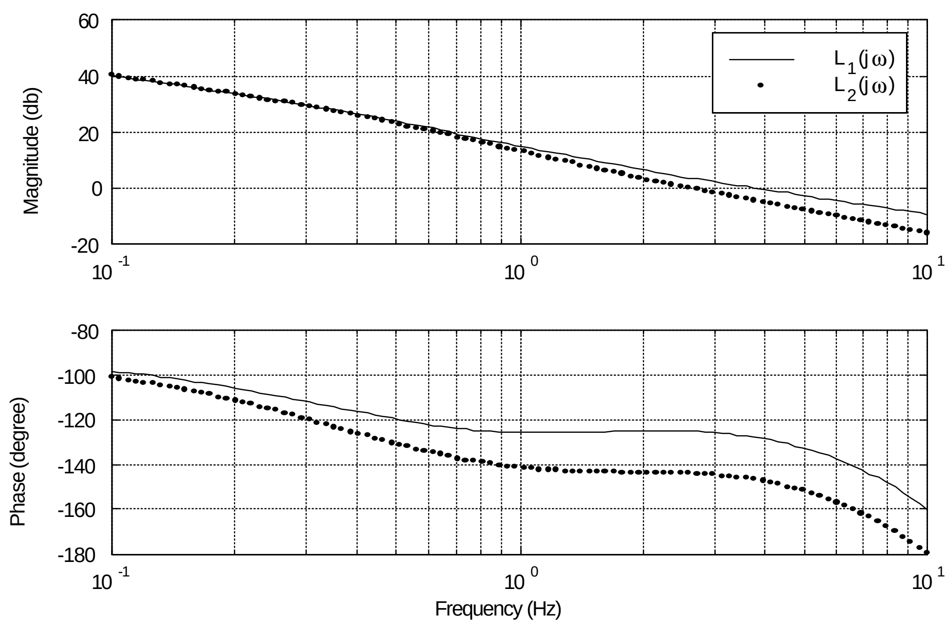

Figure 4: Bode plots of $L_1(j\omega)$ and $L_2(j\omega)$.

$\omega = 10\pi$. This specification can be met by reducing the gain of $L_1(j\omega)$ as done with $L_2(j\omega)$.

This is verified by the time response y to 10 Hz sinusoidal noise n in Figure 5. Since both designs stabilize and since both low-frequency gains are constrained by the first two specifications, Bode's gain-phase relationship [1] dictates that $L_2(j\omega)$ must have correspondingly larger phase lag as verified in the phase plot of Figure 4. The reduced gain in $L_2(j\omega)$ comes at the expense of a smaller phase margin and hence larger overshoot as shown in the step responses in Figure 6. Extensive tuning of these controllers failed to yield a design meeting all specifications. In the next subsection we synthesize a reset controller which succeeds in making the necessary tradeoffs.

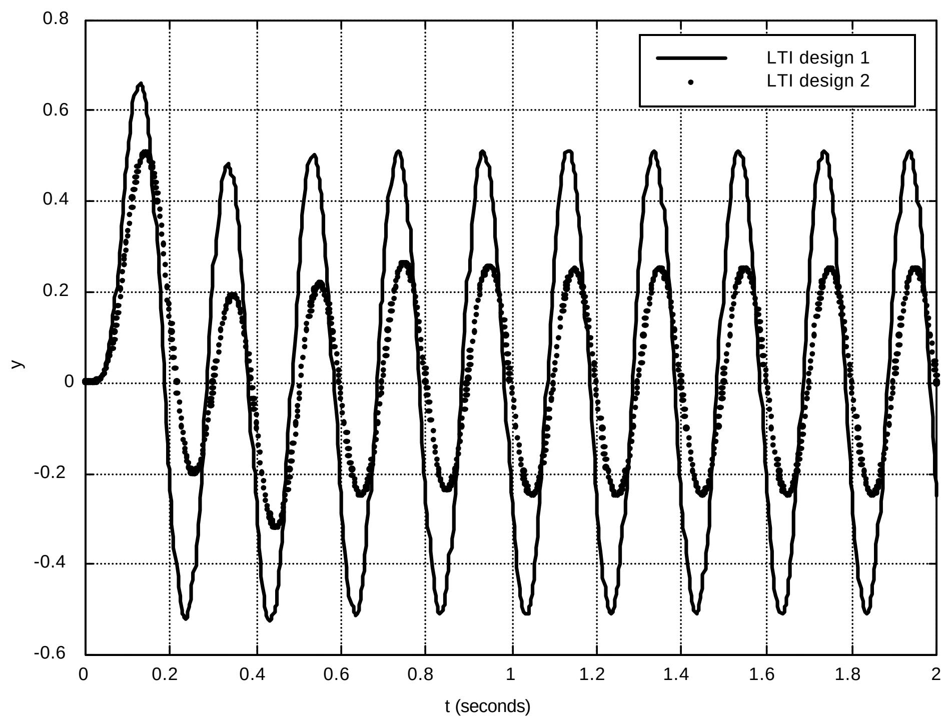

Figure 5: Comparison between LTI designs L_1 and L_2 of output response y to sensor noise $n(t) = \sin(10\pi t)$.

5.2 Reset Control Design and Analysis

Now we turn to reset control design where we exploit its potential to satisfy the above specifications. The design procedure consists of two steps as developed in [3], [5] and [6]. First, we design a linear controller to meet all the specifications - except for the overshoot constraint. For example, $C_2(s)$ is a suitable choice. The second step is to select the FORE's pole b to meet the overshoot specification. In this respect, [Figure 5, 2] provides a guideline for this choice. Using this tool, we select $b = 14$. The resulting reset control system is shown in Figure 7.

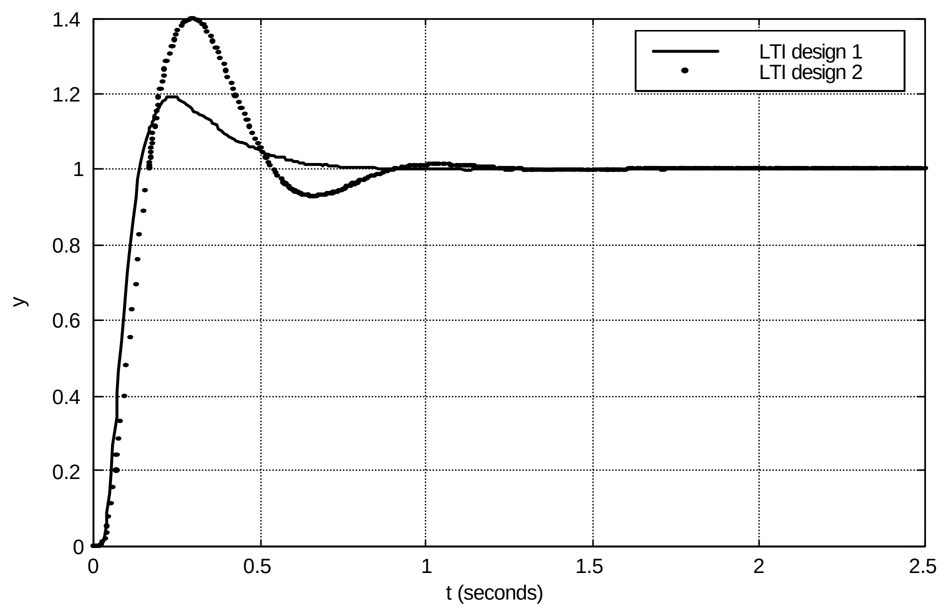

Figure 6: Comparison between LTI designs L_1 and L_2 of output response y to constant reference $r(t) \equiv 1$.

To establish stability and steady-state performance we first check the β positive-real condition (5). Since $C_2(s)$ stabilizes, then $h(s)$ in (5) is asymptotically stable. A simple search and computation shows that $\mathrm{Re}[h(j\omega)] > 0$ for all $\omega \geq 0$ when $\beta = 0.008$; see Figure 8. Invoking Theorems 1-3, we conclude that this reset control system is BIBO and asymptotically stable, and meets the asymptotic performance constraint in specification 4. Figure 9 shows an experimental result verifying the expected steady-state performance[5].

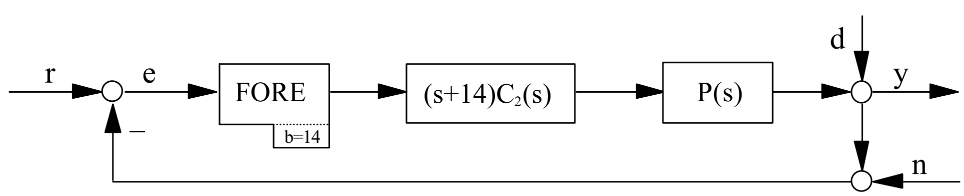

Figure 7: Reset control system for the flexible mechanism.

Finally, we compare the performance of the LTI (using L_1) and reset control systems. Figures 10 and 11 show that the reset control system has better sensor-noise suppression to a 5 Hz sinusoid and to white-noise[6]. However, unlike the LTI tradeoff experienced by controller $C_2(s)$, the reset control system has comparable transient response as shown in Figure 12.

[5]The steady-state oscillations in the step response of Figure 9 are due to tach-generator ripple.

[6]The amplification in the low-frequency spectrum in the reset control response cannot be presently explained. However, our interest is primarily in the attenuated high-frequency spectrum.

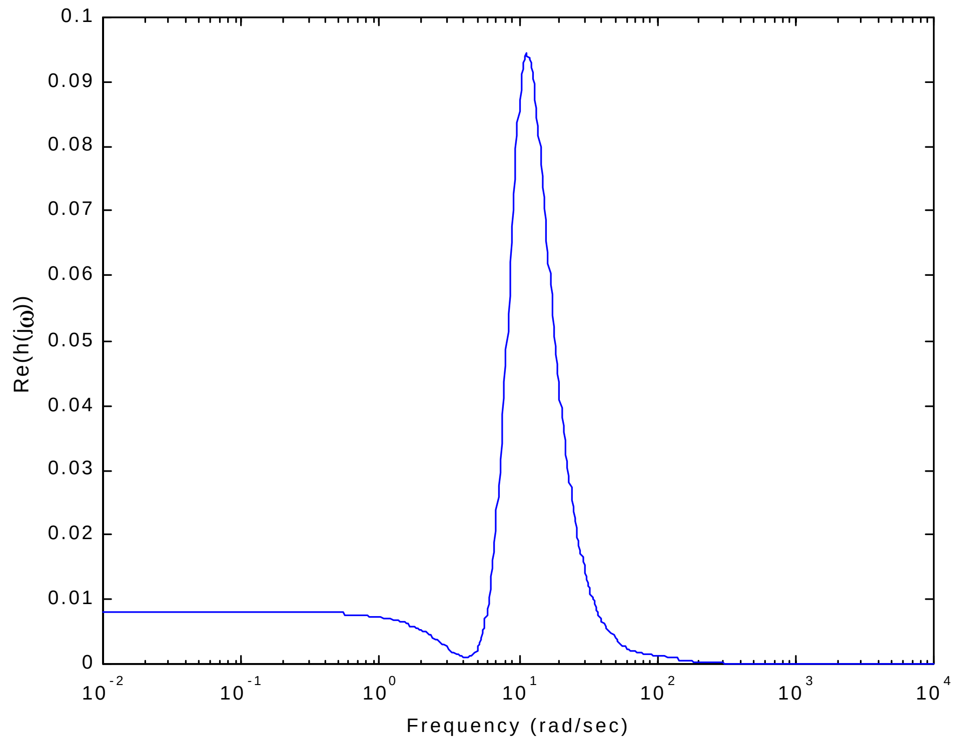

Figure 8: Plot showing $\mathrm{Re}[h(j\omega)] > 0$ when $\beta = 0.008$.

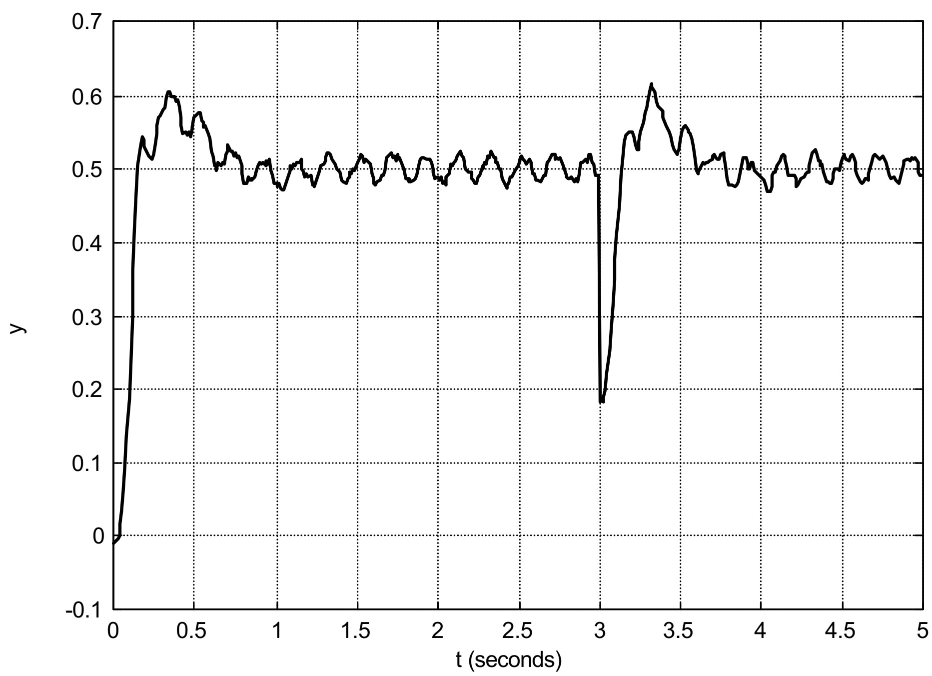

Figure 9: Output response y of reset control system to $r(t) \equiv 1; d(t) \equiv 0.5, t > 3; n(t) = -0.1e^{-t}$.

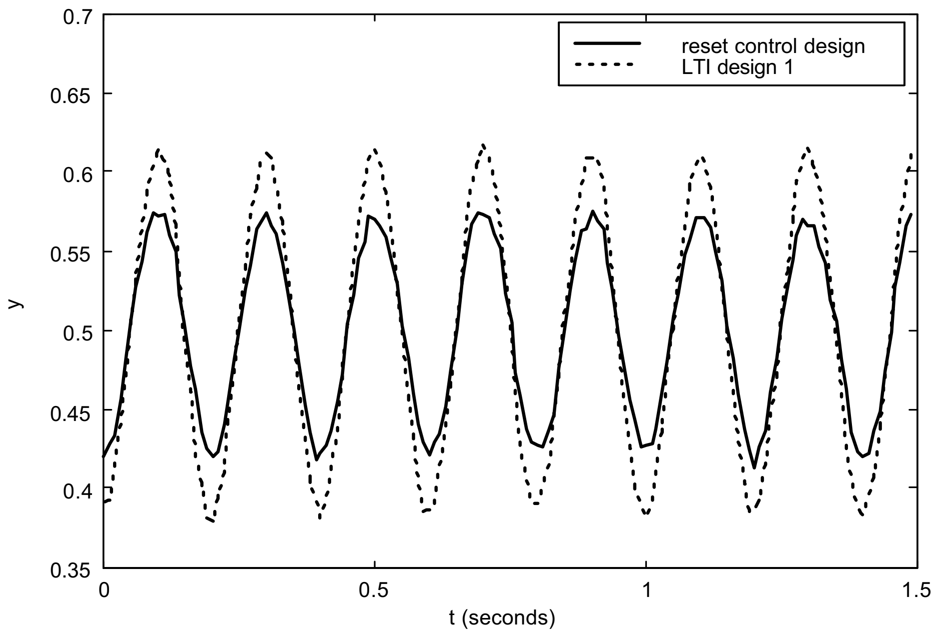

Figure 10: Comparison between reset and LTI control (using L_1) of steady-state output response y to reference $r(t) \equiv 1$ and sensor noise $n(t) = \sin(10\pi t)$.

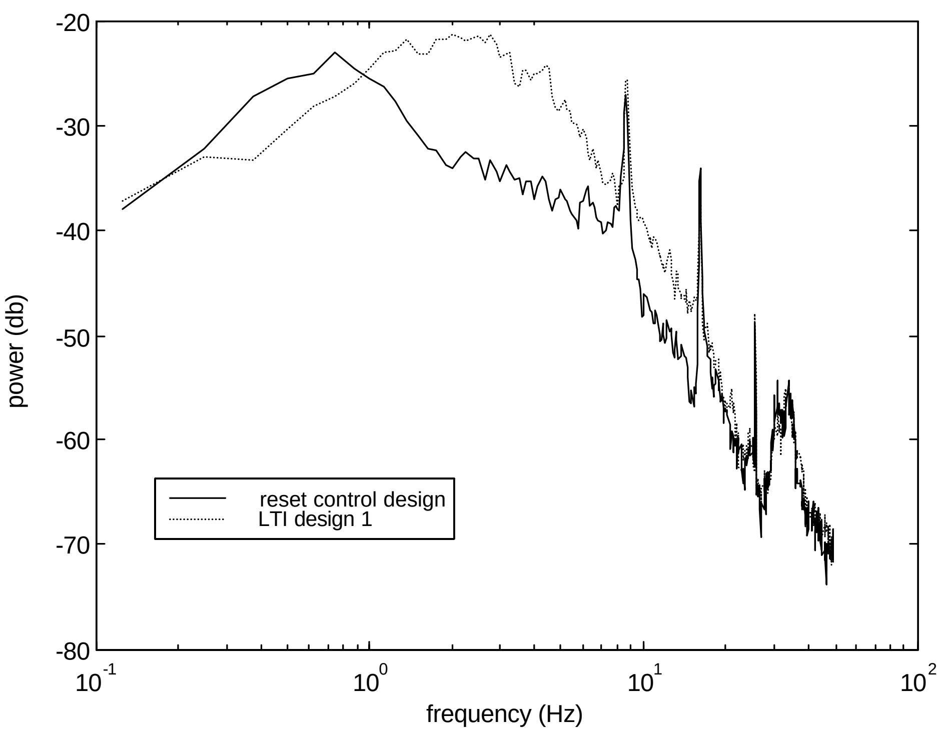

Figure 11: Comparison between reset and LTI control (using L_1) of output spectra y when n is broadband white-noise.

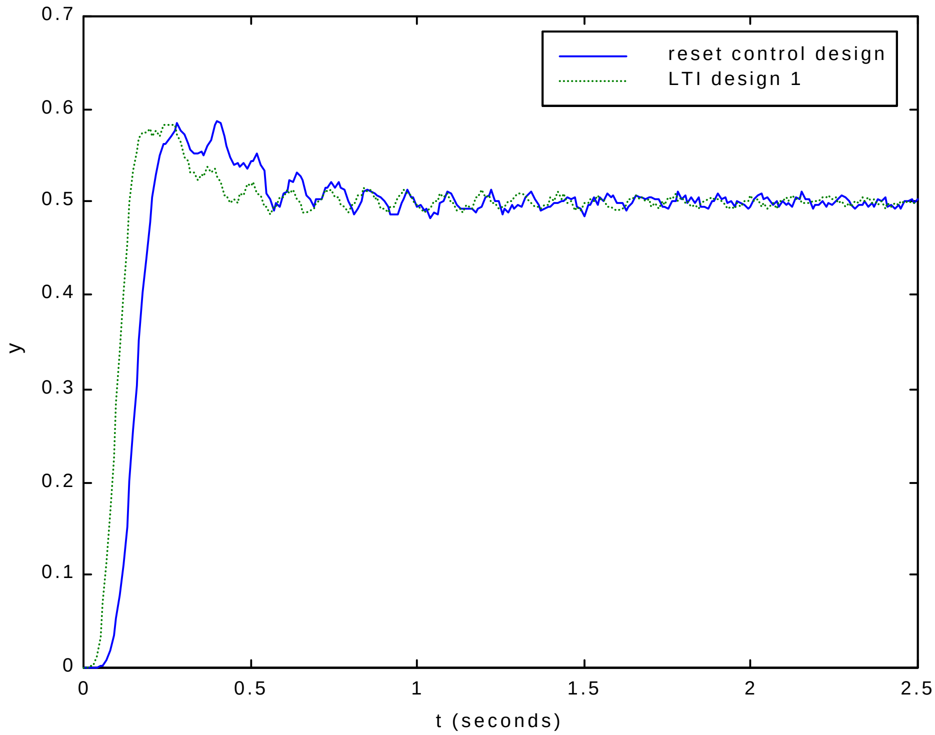

Figure 12: Comparison between reset and LTI control (using L_1) of output response y to reference $r(t) \equiv 1$.

6 Conclusion

This paper developed a sufficient condition (the β positive-real condition) for BIBO stability for a class of reset control systems. This condition also led to a series of results including asymptotic stability and steady-state performance. The β positive-real condition was shown to be satisfied in an experimental demonstration of reset control, confirming the observed performance as well as demonstrating its applicability.

Acknowledgment

The authors would like to thank Mr. Orhan Beker, a Ph.D. candidate in the ECE Department, University of Massachusetts Amherst, for many fruitful discussions.

References

[1] I.M. Horowitz, *Synthesis of Feedback Systems*, Academic Press, New York, 1963.

[2] O. Beker, C.V. Hollot and Y. Chait, "Plant with Integrator: An Example of Reset Control Overcoming Limitations of Linear Feedback," ECE Department Technical Note #ECE07.13.2000, University of Massachusetts Amherst, also submitted to *IEEE Transactions on Automatic Control*, 2000

[3] I.M. Horowitz and P. Rosenbaum, "Nonlinear Design for Cost of Feedback Reduction in Systems with Large Parameter Uncertainty," *International Journal of Control*, Vol. 24, No. 6, pp. 977-1001, 1975.

[4] P. Rosenbaum, *Reduction of the Cost of Feedback in Systems with Large Parameter Uncertainty*, Ph.D. Dissertation, Weizmann Institute of Science, Rehovot, Israel, 1977.

[5] Y. Zheng, *Theory and Practical Considerations in Reset Control Design*, Ph.D. Dissertation, University of Massachusetts, Amherst, 1998.

[6] Y. Zheng, Y. Chait, C.V. Hollot, M. Steinbuch and M. Norg, "Experimental Demonstration of Reset Control Design," *IFAC Journal of Control Engineering Practice*, Vol. 8, No. 2, pp. 113-120, 2000.

[7] J.C. Clegg, "A Nonlinear Integrator for Servomechanism," *AIEE Transactions Part II, Application and Industry*, Vol. 77, pp. 41-42, 1958.

[8] H. Hu, Y. Zheng, Y. Chait and C.V. Hollot, "On the Zero-Input Stability of Control Systems Having Clegg Integrators," *Proceedings of the American Control Conference*, pp. 408-410, Albuquerque, NM, 1997.

[9] O. Beker, C.V.Hollot, Q. Chen and Y. Chait, "Stability of A Reset Control System Under Constant Inputs," *Proceedings of the American Control Conference*, pp. 3044-3045, San Diego, CA, 1999.

[10] Q. Chen, C.V. Hollot, Y. Chait and O. Beker, "On Reset Control Systems with Second-Order Plants," *Proceedings of the American Control Conference*, pp. 205-209, Chicago, IL, 2000.

[11] Y.Z. Tsypkin, *Relay Control Systems*, Cambridge University Press, Cambridge, UK, 1984.

[12] R.A. Decarlo, "Variable Structure Control of Nonlinear Multivariable Systems: A Tutorial," *IEEE Proceedings*, Vol. 76, No. 3, pp. 212-232, 1988.

[13] M.S. Branicky, "Multiple Lyapunov Functions and Other Analysis Tools for Switched and Hybrid Systems," *IEEE Transactions on Automatic Control*, Vol. 43, pp. 475-482, 1998.

[14] D.D. Bainov and P.S. Simeonov, *Systems with Impulse Effect: Stability, Theory and Application*, Halsted Press, New York, 1989.

[15] W.M. Haddad, V. Chellabonia and N.A. Kablar, "Nonlinear Impulsive Dynamical Systems Part I: Stability and Dissipativity," *Proceedings of Conference on Decision and Control*, pp. 4404-4422, Phoenix, AZ, 1999.

[16] N.C. Singer and W.P. Seering, "Preshaping Command Inputs to Reduce System Vibration," *Transactions of the ASME*, Vol. 76, No., 3, pp. 76-82, 1990.

[17] J.E. Slotine and W. Li, *Applied Nonlinear Control*, Prentice-Hall Inc., New Jersey, 1991.

[18] Q. Chen, *Reset Control Systems: Stability, Performance and Application*, Ph.D. Dissertation, University of Massachusetts, Amherst, 2000.

[19] dSPACE Experiment Guide, dSPACE Inc., Paderborn, Germany, 1999.