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Disturbance Attenuation in a SITO Feedback Control System

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Abstract

In this paper we study disturbance attenuation problems for single-input, two-output (SITO) plants. The set of all stabilizing controllers are parametrized using two independent Youla parameters. It is then shown that tradeoffs encountered in single-loop systems can be avoided when using a second plant output for feedback.

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1 Introduction

Disturbance attenuation is a classic performance objective dating back to Bode [1] who studied the use of feedback to reduce the effect of disturbance at the plant's output. The closed-loop response to an output disturbance is governed by the sensitivity function, which can be reduced by high-gain feedback. It is now well-known that a plant 's RHP poles and zeros and its relative degree, as well as bandwidth limitations, impose constraints on sensitivity minimization [2]. Recent research on vibration and noise control applications [3] shows that improved disturbance attenuation is achieved when the measured output, used for feedback, is not collocated with the performance output. Research in [4]-[9] focused on inherent limitations of feedback systems due to physical attributes such as plant/controller structures and sensors/actuators configurations. Results were reported for a variety of applications where such limitations imposed severe constraints on closed-loop performance that could be avoided by feeding back an additional measurement. This is also related to cascade control architectures [10] commonly implemented in industrial applications; e.g., the use of inner velocity feedback in servomechanisms. Another interesting case is the inverted pendulum example in [4] where it was shown that stability margins and performance were significantly improved when both the cart position and the rod angle were fed back.

It turns out that systems studied in the references cited above can be formulated as disturbance attenuation problems having a SITO feedback structure shown in Figure 1. The SITO plant has a single input u and two outputs: the performance output z and the measured output y. The plant is then described by the two transfer functions P_{zu} and P_{yu}. The disturbance signal d affects these outputs via the disturbance transfer functions P_{zd} and P_{yd}. Our objective is to design a controller to attenuate the effect of disturbance d on performance output z.

This paper is motivated by our previous research on active noise control (ANC) in ducts [6]. A disturbance source generates noise that propagates along a duct. The goal in [6] was to design a controller, acting on a single plant measurement, to attenuate noise in the duct. We showed that limitations existed in such a single-sensor design, and, that stability margins could be improved by using a two-input single-output (TISO) control. The analysis was based on algebraic constraints of closed-loop properties at certain frequencies. However, we did not provide constructive explanation

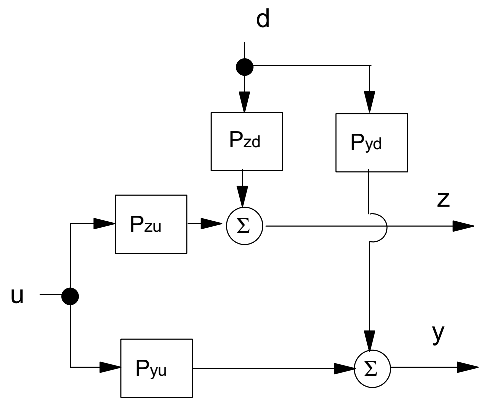


Figure 1: A SITO plant with output disturbance.

on how the additional feedback from performance microphone actually benefits the closed-loop system. The analysis in [6] focused only on the case of perfect cancellation, and the expression for the TISO case did not reveal the contribution from each controller. In this paper we address the issue with more pith by analyzing the general disturbance attenuation problem and treating perfect cancellation as a special case. Moreover, by exploiting a scheme to factorize the closed-loop system as a function of two independent parameters, we provide additional insights into the TISO controller structure that suggest a "synthesis approach" to cope with tradeoffs between performance and stability robustness. In many cases, we show that closed-loop transfer functions quantifying such tradeoffs can be expressed as a product of two factors, each with independent synthesis parameter.

The paper is organized as follows: Section 2 presents mathematical background. The main machinery used is the factorization approach developed in [11]. Section 3 formulates our disturbance attenuation problem into the standard modern control form. In Section 4 we study the parametrization of all stabilizing controllers of the SISO and TISO types. For the latter, a block diagram manipulation is proposed to transform the SITO feedback system to a two-parameters control scheme. The closed-loop transfer matrices are then described as affine functions of free parmeters ranging over all proper and stable rational functions. With this set of stable closed-loop systems we address the disturbance attenuation problem in Section 5. In Section 6 we give a simulation example by applying the analysis to the problem of ANC in ducts.

2 Preliminaries

Let $\mathbb{R}(s)$ denote the set of rational functions of a complex variable s with real coefficients. Let $\mathbf{RH}_\infty$ denote the subset of $\mathbb{R}(s)$ consisting of all proper and stable rational functions. $\mathbf{M}(\mathbf{RH}_\infty)$ is the set of matrices with elements in $\mathbf{RH}_\infty$. Recall that two polynomials $f(s)$ and $g(s)$ with real coefficients are *coprime* if their greatest common divisor is 1. We can extend this definition to two functions $f(s), g(s) \in \mathbf{RH}_\infty$. It follows that f and g are coprime (over $\mathbf{RH}_\infty$) if there exists $a, b \in \mathbf{RH}_\infty$ such that the Bezout identity

$$fa + gb = 1$$

holds. Two matrices $\mathbf{F}$ and $\mathbf{G} \in \mathbf{M}(\mathbf{RH}_\infty)$[1] are *right-coprime* if they have equal number of columns and there exists matrices $\mathbf{A}$ and $\mathbf{B} \in \mathbf{M}(\mathbf{RH}_\infty)$ such that

$$[\mathbf{A}\ \ \mathbf{B}] \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} = \mathbf{AF} + \mathbf{BG} = \mathbf{I}.$$

Likewise, two matrices $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{B}} \in \mathbf{M}(\mathbf{RH}_\infty)$ are *left-coprime* if they have equal number of rows and there exists matrices $\mathbf{A}$ and $\mathbf{B} \in \mathbf{M}(\mathbf{RH}_\infty)$ such that

$$[\tilde{\mathbf{F}}\ \ \tilde{\mathbf{G}}] \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} = \tilde{\mathbf{F}}\mathbf{A} + \tilde{\mathbf{G}}\mathbf{B} = \mathbf{I}.$$

3 Problem Statement

The disturbance attenuation problem described in Section 1 and shown in Figure 1 can be formulated into the standard configuration shown in Figure 2 where

$$\mathbf{G} = \begin{bmatrix} G_{zd} & G_{zu} \\ G_{yd} & G_{yu} \end{bmatrix}$$

is the generalized plant and $\mathbf{C}$ is the controller to be synthesized. The input to $\mathbf{C}$ is either the measured output y (SISO case) or both outputs y and z (TISO case). Let G_{zd}, G_{zu}, G_{yd} and G_{yu}

[1] Matrices are denoted using boldface font.

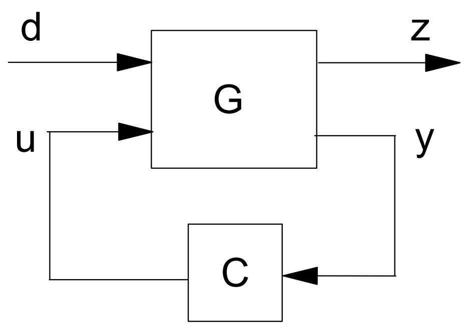


Figure 2: A modern control design paradigm.

denote the transfer functions between the indicated inputs and outputs. Since $G_{zd} = P_{zd}$ and $G_{zu} = P_{zu}$, the closed-loop transfer function from disturbance input d to performance output z is

$$T_{zd} = P_{zd} + P_{zu}\mathbf{C}(\mathbf{I} - \mathbf{G}_{yu}\mathbf{C})^{-1}\mathbf{G}_{yd}. \tag{1}$$

In general design specifications are frequency dependent. For example, in ANC applications we usually deal with low-frequency disturbance signals. In such case an appropriate weighting function $W_p(s)$ can be defined. The performance objective can then be stated as

$$|W_p(j\omega)T_{zd}(j\omega)| \leq 1, \quad \forall\omega. \tag{2}$$

This is the simplest description for the disturbance attenuation problem where our only concern is to minimize the closed-loop response to disturbance. In real applications it is impractical to disregard other issues such as stability robustness or limitations on control actuator effort. These supplementary objectives may be included in the performance output (vector) z.

4 Parametrization of All Stabilizing Controllers

In this section we apply the concept of of parametrization of all stabilizing controllers from [11] to our problem. We begin our analysis with the case of using only measurement output y as feedback.

4.1 SISO Controller

Consider the SISO feedback system shown in Figure 3. Let d_i denote an input disturbance signal while d_z and d_y are output disturbances. Since P_{zd} and P_{yd} are outside the feedback loop, they cannot destabilize the closed-loop system. The system in Figure 3 is then described by

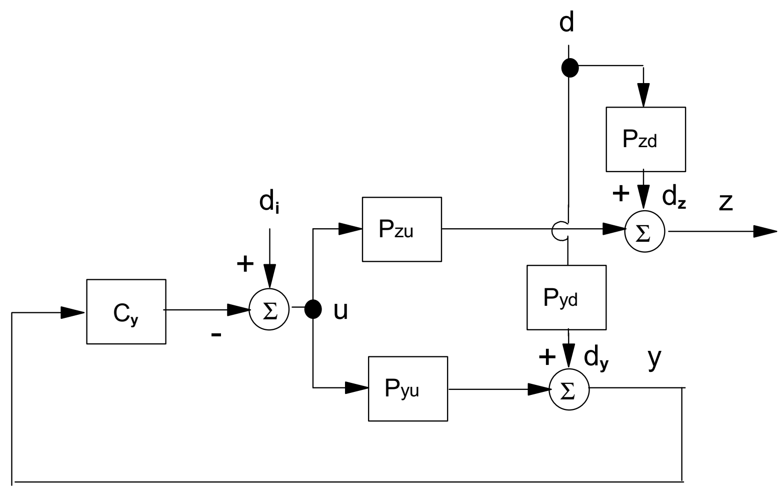


Figure 3: Disturbance attenuation using only measurement output y.

$$\begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} d_z \\ d_y \\ d_i \end{bmatrix} + \begin{bmatrix} 0 & 0 & P_{zu} \\ 0 & 0 & P_{yu} \\ 0 & -C_y & 0 \end{bmatrix} \begin{bmatrix} z \\ y \\ u \end{bmatrix}, \tag{3}$$

which yields

$$\begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} 1 & \frac{-P_{zu}C_y}{1+P_{yu}C_y} & \frac{P_{zu}}{1+P_{yu}C_y} \\ 0 & \frac{1}{1+P_{yu}C_y} & \frac{P_{yu}}{1+P_{yu}C_y} \\ 0 & \frac{-C_y}{1+P_{yu}C_y} & \frac{1}{1+P_{yu}C_y} \end{bmatrix} \begin{bmatrix} d_z \\ d_y \\ d_i \end{bmatrix}. \tag{4}$$

Define

$$\mathbf{H}(P,C) \triangleq \begin{bmatrix} 1 & \frac{-P_{zu}C_y}{1+P_{yu}C_y} & \frac{P_{zu}}{1+P_{yu}C_y} \\ 0 & \frac{1}{1+P_{yu}C_y} & \frac{P_{yu}}{1+P_{yu}C_y} \\ 0 & \frac{-C_y}{1+P_{yu}C_y} & \frac{1}{1+P_{yu}C_y} \end{bmatrix}. \tag{5}$$

Then, C_y stabilizes P_{yu} if and only if $\mathbf{H}(P,C) \in \mathbf{M}(\mathbf{RH}_\infty)$. Now we state without proof a theorem from [11], which provides a parametrization of all controllers that stabilize $\mathbf{H}(P,C)$.

Theorem 1: *Suppose $P_{yu} \in \mathbb{R}(s)$. Factorize P_{yu} as $n_{yu}d_{yu}^{-1}$ where $n_{yu}d_{yu} \in \mathbf{RH}_\infty$ are coprime. Select $a, b \in \mathbf{RH}_\infty$ such that $an_{yu} + bd_{yu} = 1$. Then, the set of all compensators that stabilize P_{yu}, denoted by $\mathbf{S}(P_{yu})$, is given by*

$$\mathbf{S}(P_{yu}) = \left\{ C_y = \frac{a + rd_{yu}}{b - rn_{yu}} : r \in \mathbf{RH}_\infty \ \ \textit{and} \ \ b - rn_{yu} \neq 0 \right\}. \tag{6}$$

Applying this controller description to (4) yields

$$\begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} 1 & -P_{zu}d_{yu}(a + rd_{yu}) & P_{zu}d_{yu}(b - rn_{yu}) \\ 0 & d_{yu}(b - rn_{yu}) & n_{yu}(b - rn_{yu}) \\ 0 & -d_{yu}(a + rd_{yu}) & d_{yu}(b - rn_{yu}) \end{bmatrix} \begin{bmatrix} d_z \\ d_y \\ d_i \end{bmatrix}. \tag{7}$$

The net effect of Theorem 1 is that, both the set of all stabilizing controllers and the corresponding closed-loop system matrices can be parameterized in terms of a single "free" parameter $r \in \mathbf{RH}_\infty$ which ranges over all proper and stable transfer functions. Moreover, the closed-loop transfer matrices $\mathbf{H}(P, C)$ become an affine function of this parameter.

4.2 TISO controller

Now consider the case when both the measurement and performance outputs are used as feedback as shown in Figure 4. We have a second controller C_z closing the loop from performance output z.

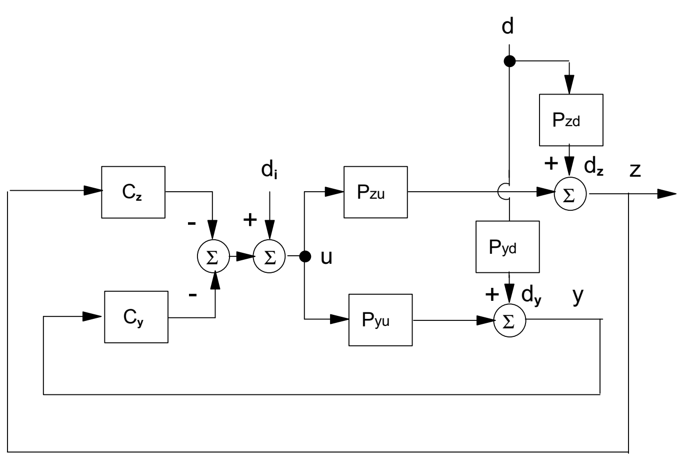


Figure 4: Disturbance attenuation using a TISO controller.

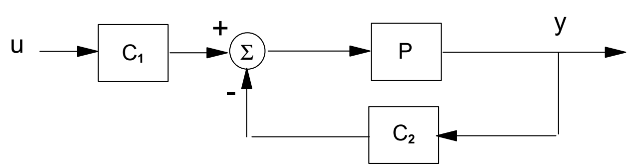
With some straightforward calculation we can describe the closed-loop system as

$$\begin{bmatrix} z \\ y \\ u \end{bmatrix} = \mathbf{W}(P, C) \begin{bmatrix} d_z \\ d_y \\ d_i \end{bmatrix} \tag{8}$$

where

$$\mathbf{W}(P, C) \triangleq \frac{1}{1 + P_{yu}C_y + P_{zu}C_z} \begin{bmatrix} 1 + P_{yu}C_y & -P_{zu}C_y & P_{zu} \\ -P_{yu}C_z & 1 + P_{zu}C_z & P_{yu} \\ -C_z & -C_y & 1 \end{bmatrix}.$$

We want to represent the closed-loop system as a function of two free parameters. One way to do so is to manipulate the block diagram in Figure 4 into the two-degree-of-freedom scheme shown in Figure 5(a). To handle unstable C_1, we use the representation suggested in [11] by letting $(\tilde{d}_c, [\tilde{n}_{c1} \quad \tilde{n}_{c2}])$ be a left-coprime factorization of $\mathbf{C} = [C_1 \quad C_2]$. Then, with the factorization of $P = n_p d_p^{-1}$ the block diagrams in Figure 5(a) and Figure 5(b) are equivalent. We will exploit the latter in our analysis.



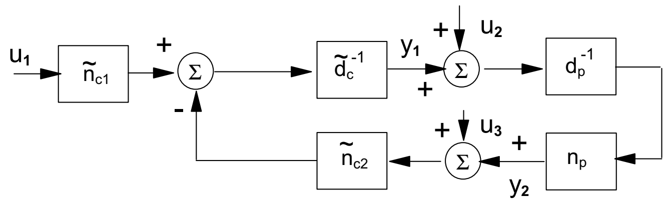


Figure 5: A two-parameter control scheme.

Proposition 1: *The two-parameter scheme in Figure 5(a) with* $\mathbf{C} = [C_1 \quad C_2]$ *can be represented by the block diagram in Figure 5(b), with* $(\tilde{d}_c, [\tilde{n}_{c1} \quad \tilde{n}_{c2}])$ *left-coprime.*

Proof: For any C_1 and C_2, we can find their coprime factorizations as $C_1 = d_1^{-1} n_1$ and $C_2 = d_2^{-1} n_2$, with $n_1, n_2, d_1, d_2 \in \mathbf{RH}_\infty$ and d_1, d_2 proper. Select $\tilde{d}_c$ as a least common multiple of d_1 and d_2. We factorize d_1 and d_2 as $d_1 = CB$ and $d_2 = CA$ such that C contains all common zeros of d_1 and d_2. Then $\tilde{d}_c = Ad_1 = Bd_2$ is a least common multiple of d_1 and d_2. Note that A and B are coprime, since by construction they have no common zeros in the extended RHP[2] [11]. Then, we select $[\tilde{n}_{c1} \;\; \tilde{n}_{c2}] = [An_1 \;\; Bn_2]$.

Now we show that $(\tilde{d}_c, [An_1 \;\; Bn_2])$ is left-coprime. From the coprimeness of $(n_1, d_1), (n_2, d_2)$ and (A, B) there exist $D, E, F, G, X, Y \in \mathbf{RH}_\infty$ such that $Dn_1 + Ed_1 = 1, Fn_2 + Gd_2 = 1$, and $XA + YB = 1$. Together with $\tilde{d}_c = Ad_1 = Bd_2$, we have

$$XADn_1 + XE\tilde{d}_c = XA$$

$$YBFn_2 + YG\tilde{d}_c = YB.$$

It is then easy to verify that

$$[An_1 \;\; Bn_2] \begin{bmatrix} XD \\ YF \end{bmatrix} + \tilde{d}_c(XE + YG) = XA + YB = 1.$$

Hence $(\tilde{d}_c, [An_1 \;\; Bn_2])$ are left-coprime. □

Now we state a scalar version of Theorem 5.6.15 in [11].

Theorem 2: *Consider the two-parameter scheme in Figure 5(b). Let (d_p, n_p) be a coprime factorization of P and $(\tilde{d}_c, [\tilde{n}_{c1} \;\; \tilde{n}_{c2}])$ be a left-coprime factorization of $\mathbf{C} = [C_1 \;\; C_2]$. Select $a, b \in \mathbf{RH}_\infty$ such that $an_p + bd_p = 1$. Then the set of all two-parameter compensators that stabilize P is given by*

$$\mathbf{S}_2(P) = \left\{ \mathbf{C} = (b - rn_p)^{-1}[q \;\; a + rd_p] : q, r \in \mathbf{RH}_\infty \;\; \text{and} \;\; b - rn_p \neq 0 \right\}. \tag{9}$$

The set of all possible stable transfer matrices from (u_1, u_2, u_3) to (y_1, y_2) in Figure 5(b) is represented by

$$\begin{bmatrix} d_p q & d_p(b - rn_p) - 1 & -d_p(a + rd_p) \\ n_p q & n_p(b - rn_p) & -n_p(a + rd_p) \end{bmatrix}. \tag{10}$$

[2] That is, closed right half-plane together with the point at infinity.

Proof: The proof is similar to [11] so only a sketch is included here for reference. The theorem is based on the following Lemma.

Lemma 1: *Let (n_p, d_p) be a coprime factorization of P and let $(\tilde{d}_c, [\tilde{n}_{c1} \;\; \tilde{n}_{c2}])$ be a left-coprime factorization of $C = [C_1 \;\; C_2]$. Then, the system of Figure 5(b) is stable if and only if*

$$\tilde{d}_c d_p + \tilde{n}_{c2} n_p = k \;\; (constant). \tag{11}$$

Then, from Theorem 1, the set of all stabilizing controller $\tilde{d}_c^{-1}\tilde{n}_{c2}$ equal $(b - rn_p)^{-1}(a + rd_p)$ for $r \in \mathbf{RH}_\infty$ such that $(b - rn_p) \neq 0$. Since Lemma 1 shows that closed-loop stability cannot be affected by $\tilde{n}_{c1}$, we are allowed to choose $\tilde{n}_{c1}$ as another free parameter $q \in \mathbf{RH}_\infty$. This gives immediately the set $\mathbf{S}_2(P)$ in (9). (10) follows by direct computation. $\square$

Theorem 2 states that the transfer matrix from (u_1, u_2, u_3) to (y_1, y_2) involves two independent parameters q and r. In the sequel we will show how this additional degree of freedom helps in our disturbance attenuation problem. In order to do so, we need to transform our SITO feedback system to an equivalent form as in Figure 5(b). The following corollary can help simplify the process.

Corollary 1: *Let (n_1, d_1) and (n_2, d_2) be coprime factorizations of the controller C_1 and C_2 in Theorem 2. Then the two-parameter scheme cannot stabilize P unless d_2 is a multiple of d_1.*

Proof: A necessary condition for (11) in Lemma 1 is that $\tilde{d}_c$ and $\tilde{n}_{c2}$ are coprime. From the constructive proof of Proposition 1, it follows that $\tilde{d}_c = Bd_2$ and $\tilde{n}_{c2}$ are coprime if and only if B is a constant. Hence $\tilde{d}_c = Ad_1 = d_2$ must be a least common multiple of d_1 and d_2. This implies d_2 is a multiple of d_1. $\square$

Remark 1: We can conclude from Corollary 1 that, for the two-parameter controller $\mathbf{C} = [C_1 \;\; C_2]$ to stabilize P, every RHP pole of C_1 must also be a pole of C_2 with at least the same multiplicity.

We are now ready to transform our SITO feedback system into an equivalent two-parameter form. The manipulation steps are shown in Figure 6. Step (a) to (b) is a straightforward block-diagram reduction. Note that in step (b) we have a two-parameter representation identical to

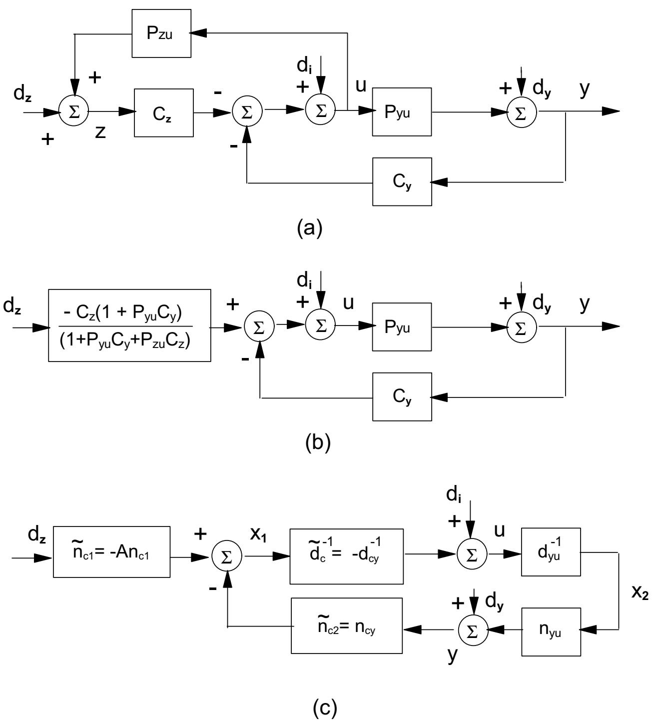


Figure 6: Block diagram manipulation steps to a 2-parameter form of Figure 5(b).

that of Figure 5(a), with $\mathbf{C}^{\dagger} = [C_1 \quad C_2] = \left[-\frac{C_z(1+P_{yu}C_y)}{1+P_{yu}C_y+P_{zu}C_z} \quad C_y\right]$. Let (n_{c1}, d_{c1}) and (n_{cy}, d_{cy}) be coprime factorizations of C_1 and C_y, respectively. Using the method suggested in the proof of Proposition 1, we can construct $\mathbf{C}^{\dagger} = \tilde{d}_c^{-1}[\tilde{n}_{c1} \quad \tilde{n}_{c2}]$ with $\tilde{d}_c = Ad_1 = Bd_{cy}$ as a least common multiple of d_1 and d_{cy}, and with $[\tilde{n}_{c1} \quad \tilde{n}_{c2}] = [-An_{c1} \quad Bn_{cy}]$. However, Corollary 1 implies that B should be a constant. Without loss of generality we set $B = 1$. Hence the final two-parameter scheme in Figure 6(c) is represented by $\mathbf{C}^{\dagger} = \tilde{d}_c^{-1}[\tilde{n}_{c1} \;\; \tilde{n}_{c2}]$, with

$$\tilde{d}_c = d_{cy}; \quad [\tilde{n}_{c1} \;\; \tilde{n}_{c2}] = [-An_{c1} \quad n_{cy}].$$

From Proposition 1, $(\tilde{d}_c, [\tilde{n}_{c1} \;\; \tilde{n}_{c2}])$ are left-coprime.

Theorem 2 can now be invoked to yield the set of all stabilizing controllers $\mathbf{C}^{\dagger}$ represented by (9). With some routine calculations, the set of all stabilizing controllers $\mathbf{C} = [C_z \;\; C_y]$ in our original setup is found to be

$$C_z \quad = \quad \frac{q}{(b - rn_{yu})(1 - qd_{yu}P_{zu})};$$

$$C_y = \frac{a + r d_{yu}}{b - r n_{yu}} \tag{12}$$

provided that $(b - r n_{yu}) \neq 0, (1 - q d_{yu} P_{zu}) \neq 0$. With this set of controllers, the closed-loop transfer matrix from (d_z, d_y, d_i) to (z, y, u) in (8) becomes

$$\begin{bmatrix} z \\ y \\ u \end{bmatrix} = \mathbf{W}(P, C) \begin{bmatrix} d_z \\ d_y \\ d_i \end{bmatrix} \tag{13}$$

with

$$\mathbf{W}(P, C) = \begin{bmatrix} 1 - P_{zu} d_{yu} q & -P_{zu} d_{yu}(a + r d_{yu})(1 - P_{zu} d_{yu} q) & P_{zu} d_{yu}(b - r n_{yu})(1 - P_{zu} d_{yu} q) \\ -n_{yu} q & d_{yu}(b - r n_{yu})(1 - P_{zu} d_{yu} q) + P_{zu} d_{yu} q & n_{yu}(b - r n_{yu})(1 - P_{zu} d_{yu} q) \\ -d_{yu} q & -d_{yu}(a + r d_{yu})(1 - P_{zu} d_{yu} q) & d_{yu}(b - r n_{yu})(1 - P_{zu} d_{yu} q) \end{bmatrix}.$$

5 Disturbance attenuation and stability robustness

The analysis in the last section allows us to compare SISO and TISO controllers. We will demonstrate that the tradeoffs between attenuation performance and stability margins, normally difficult to overcome by a SISO design, can be alleviated by the additional degree of freedom provided by a TISO controller.

Recall from the problem statement in Section 3 that our objective is to minimize T_{zd}, the transfer function from output disturbance d to performance output z. We are less interested in the input disturbance so we set $d_i = 0$. Furthermore, in this paper we concentrate only on the case of stable plant P_{yu}. Suppose P_{yu} is stable, then we can choose $(n_{yu}, d_{yu}) = (P_{yu}, 1)$ as a coprime factorization and take $a = 0, b = 1$ to satisfy the Bezout identity $a n_{yu} + b d_{yu} = 1$. Substituting into (6),(7),(12), and (13) and removing the elements that correspond to d_i, we have the following descriptions for controllers and closed-loop systems:

SISO case:

$$C_y = \frac{r}{1 - r P_{yu}}; \tag{14}$$

$$\begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} 1 & -rP_{zu} \\ 0 & 1-rP_{yu} \\ 0 & -r \end{bmatrix} \begin{bmatrix} d_z \\ d_y \end{bmatrix}. \tag{15}$$

TISO case:

$$\begin{aligned} C_z &= \frac{q}{(1-rP_{yu})(1-qP_{zu})}; \\ C_y &= \frac{r}{1-rP_{yu}}; \end{aligned} \tag{16}$$

$$\begin{bmatrix} z \\ y \\ u \end{bmatrix} = \begin{bmatrix} 1-qP_{zu} & -rP_{zu}(1-qP_{zu}) \\ -qP_{yu} & (1-rP_{yu})(1-qP_{zu})+qP_{zu} \\ -q & -r(1-qP_{zu}) \end{bmatrix} \begin{bmatrix} d_z \\ d_y \end{bmatrix}. \tag{17}$$

By noting that $d_z = P_{zd}d$ and $d_y = P_{yd}d$, we have the expressions for T_{zd} as

SISO case:

$$T_{zd-SISO} = P_{zd} - P_{zu}rP_{yd} \tag{18}$$

TISO case:

$$T_{zd-TISO} = (P_{zd} - P_{zu}rP_{yd})(1-qP_{zu}). \tag{19}$$

Disturbance attenuation for the SISO controller case then reduces to a standard model-matching problem [12]; that is, given $P_{zd}, P_{zu}, P_{yd} \in \mathbf{RH}_\infty$, find $r \in \mathbf{RH}_\infty$ such that $||P_{zd} - P_{zu}rP_{yd}||_\infty$ is minimized. In the TISO controller case we have another factor $(1-qP_{zu})$ in the expression, which depends solely on the second free parameter $q \in \mathbf{RH}_\infty$.

Consider a more practical problem where we not only want to have a good disturbance attenuation performance, but also require that the closed-loop system is robust to uncertainties like modeling and measurement errors. Using standard unstructured uncertainty analysis [12], it can be shown that stability robustness to multiplicative uncertainty in P_{yu} can be quantified by the H_∞ norm of the complementary sensitivity transfer function in the output y loop. Since the element (2,2) in the closed-loop transfer matrices of (15) and (17) are the sensitivity transfer functions from d_y to y, denoted by $S_y(s)$. The complementary sensitivity $T_y(s)$ then equals $1 - S_y(s)$; i.e.,

SISO case:

$$T_{y-SISO} = rP_{yu} \tag{20}$$

TISO case:

$$T_{y-TISO} = rP_{yu}(1 - qP_{zu}). \tag{21}$$

Suppose we want to have strong disturbance attenuation and stability robustness to multiplicative uncertainty in P_{yu}. This translates to simultaneous minimization of $||T_{zd}||_\infty$ and $||T_y||_\infty$. We demonstrate the difficulty of such problem in the special case of perfect cancellation $||T_{zd}||_\infty = 0$.

5.1 Perfect Cancellation of Output Disturbances

By perfect cancellation we refer to the case where the performance output response z from d_z and d_y are exactly zero. Even though this situation cannot occur in practice, we study it to gain insight into the benefit from using the performance output as feedback. From (18) and (19), perfect cancellation can be achieved if we can find $r \in \mathbf{RH}_\infty$ such that

$$r = \frac{P_{zd}}{P_{zu}P_{yd}}. \tag{22}$$

In the SISO case, substituting this r into (20) gives

$$T_{y-SISO} = \frac{P_{zd}P_{yu}}{P_{zu}P_{yd}}. \tag{23}$$

Hence $||T_y||_\infty$ depends solely on the magnitude of open-loop plant and disturbance transfer functions. However, when using the TISO controller, substituting r in (22) to (21) yields

$$T_{y-TISO} = \frac{P_{zd}P_{yu}}{P_{zu}P_{yd}}(1 - qP_{zu}). \tag{24}$$

The additional free parameter q can be used to favorably affect $||T_y||_\infty$.

6 Example: Active Noise Control

To provide a concrete example, we apply our analysis to an ANC problem in an acoustic duct. This disturbance attenuation application remains a challenging control problem. In theory the duct dynamics represents a distributed-parameter system consisting of infinite numbers of lightly-damped modes. We have shown in [7] the tradeoff between stability and performance is sensitive to sensor and actuator placements. One such configuration is intentionally selected in this example to appreciate the benefit of using a TISO controller.

Consider the acoustic duct configuration in Figure 7. The speaker located upstream simulates a

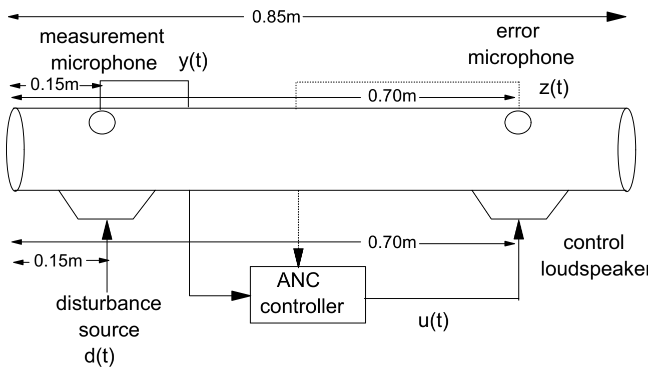


Figure 7: ANC in an acoustic duct.

disturbance source that injects acoustic "noise" into the duct. A measurement microphone detects the disturbance near the source and the downstream error microphone measures the level of noise cancellation at a point in the duct where noise attenuation is desired. The objective of the controller is to minimize the acoustic energy at the error microphone.

We formulate this duct ANC problem using our SITO framework as follows. Let $P_{zd}(s)$, $P_{zu}(s)$, $P_{yd}(s)$, and $P_{yu}(s)$ denote the transfer functions from the disturbance speaker to the error microphone, the control speaker to the error microphone, the disturbance speaker to the measurement microphone, and the control speaker to the measurement microphone, respectively. In an SISO design scheme, C_y represents the controller that uses information from the measurement microphone y as input. When a TISO controller is used, the performance output z from the error microphone is fed back to another controller C_z[3]. Using any H_∞ software the TISO controller can be synthesized in one step. To be consistent with the analysis in this paper, however, we first perform one-sensor design and assess closed-loop performance and stability margins from C_y acting alone. We then design another free parameter q such that the resulting TISO controller $\mathbf{C} = [C_z \quad C_y]$

[3]In ANC literature, C_y and C_z are usually referred to as *feedforward* and *feedback* controllers, respectively.

yields satisfactory improvement in stability robustness.

We use the same acoustic duct and speaker models as in [7], with the duct length and microphone/speaker locations shown in Figure 7. Six modes are retained. See [8] for more details. The problem data is casted into a standard H_∞ setup described in Section 3. For simplicity the performance weight W_p in (2) is chosen as constant. Our primary objective is to achieve the best possible attenuation performance; i.e., to minimize T_{zd} in (1). For robust stability sake, another weight $W_s = 0.05\left(\frac{1}{2\pi 600} + 1\right)^2 \left(\frac{1}{20000} + 1\right)^{-2}$ is applied to guard against additive uncertainty in P_{mu}. The resulting controller C_y yields frequency responses in Figure 8. The top figure depicts disturbance

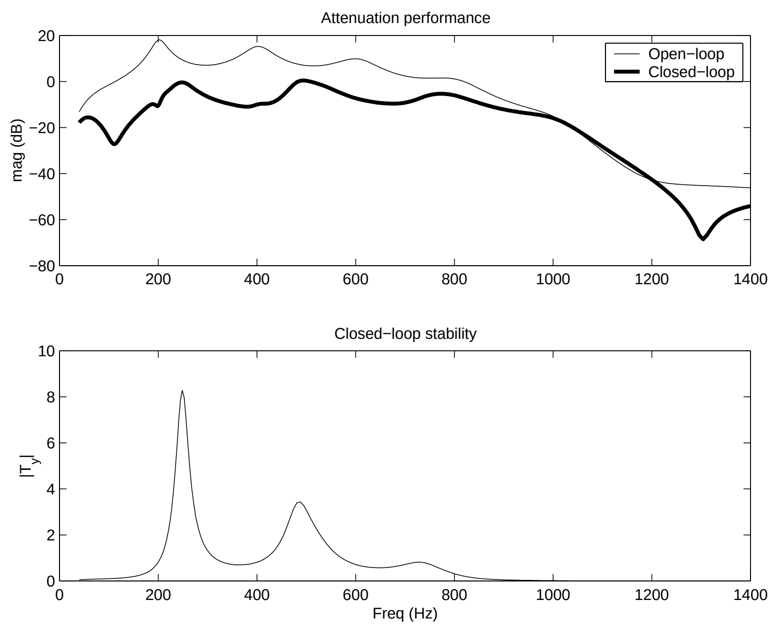


Figure 8: Frequency response from one-sensor design.

attenuation performance by comparing the open-loop and closed-loop responses at output z, while the magnitude of complementary sensitivity T_y is plotted in the bottom figure. These plots are consistent with those from experimental data given in [6]. They reveal the fact that high peaking in $|T_y(j\omega)|$ should occur at some frequency in this particular sensor/actuator configuration [7]. In this example, the peak is most pronounced at 250 Hz, with $||T_y||_\infty = |T_y(j2\pi 250)| = 8.5$.

In the SITO structure, from (20) and (21) we see that there is an additional free parameter q via the feedback controller C_z. Here we employ an approach by designing q directly such that

$||1 - qP_{zu}||_\infty \ll 1$.[4] With this q and the parameter r computed from C_y, we solve for C_z using (16). Figure 9 shows the performance comparison at output z. We see some slight improvement of attenuation level after adding C_z. This can be explained from (19) where the minimization of

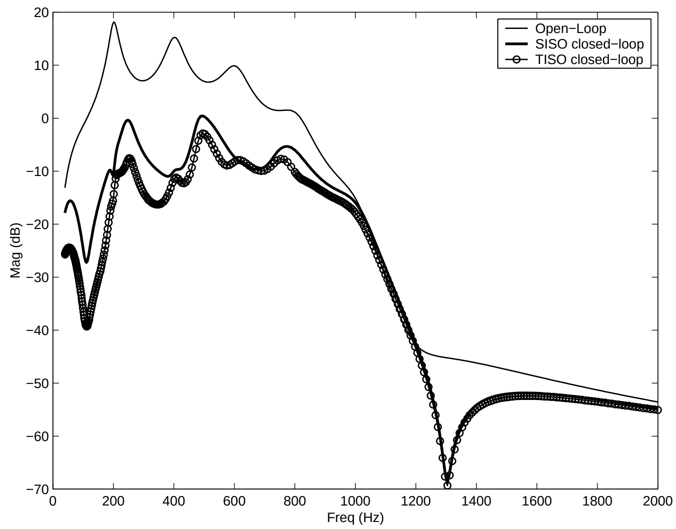


Figure 9: Performance comparison between SISO and TISO controllers.

$||1 - qP_{zu}||_\infty$ also helps to reduce $||T_{zd}||_\infty$. We have found, however, that too much reduction on $||1 - qP_{zu}||_\infty$ not only renders C_z too aggressive but also tends to make C_z unstable. In general unstable controllers are difficult to implement because of their limited gain-reduction margins and the inability to perform open-loop tests [14].

Let us now evaluate closed-loop stability margins. Assume as before that all plant perturbations are described by multiplicative uncertainties, a measure of stability is then related to the size of complementary sensitivity transfer functions. We should note that for the TISO controller case the closed-loop system consists of two feedback paths via C_z and C_y, so that suitable transfer functions to be compared are $\mathbf{T}_O(s)$, the output complementary sensitivity. Using (15) and (17), and the algebraic constraint $\mathbf{S}_O + \mathbf{T}_O = \mathbf{I}, \mathbf{T}_O(s)$ is found to be

[4]Since the factor $(1 - qP_{zu})$ appears in the denominator of C_z, too small $||1 - qP_{zu}||_\infty$ may cause an excessively-large actuator effort. This can be prevented by adding a constraint on $||q||_\infty$. See [8] for details.

SISO case:

$$\mathbf{T}_{O-SISO} = \begin{bmatrix} 0 & rP_{zu} \\ 0 & rP_{yu} \end{bmatrix} \tag{25}$$

TISO case:

$$\mathbf{T}_{O-TISO} = \begin{bmatrix} qP_{zu} & rP_{zu}(1 - qP_{zu}) \\ qP_{yu} & rP_{yu}(1 - qP_{zu}) \end{bmatrix} \tag{26}$$

Figure 10 shows the magnitude frequency response comparison of (25) and 26). We observe that $||\mathbf{T}_O||_\infty = |T_{O22}(j2\pi 250)|$ is reduced from 8.5 to 4 by the TISO controller.

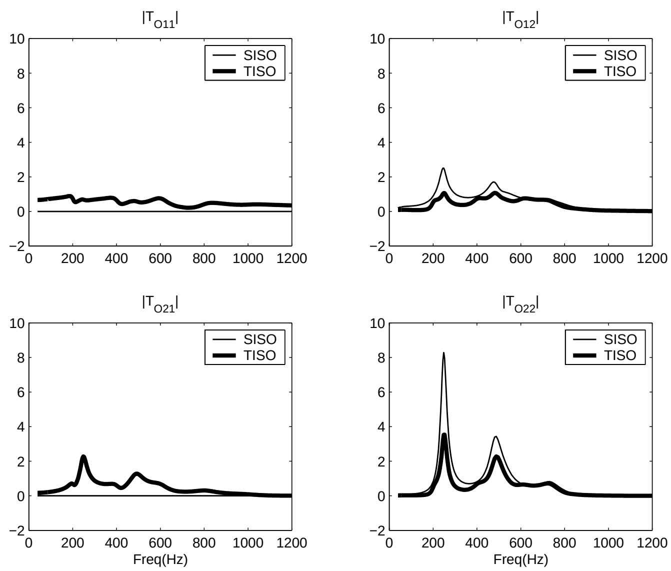


Figure 10: Comparison of output complementary sensitivity $|\mathbf{T}_O|$.

Remark 2: It is interesting to note a slight increase in $|T_{O21}|$ with the TISO controller. This gives another reason why a good design may require an additional constraint on $||q||_\infty$ [8].

7 Conclusions

This paper developed an in-depth analysis of disturbance attenuation problem on a SITO plant. Our analysis suggests that using both measurement and performance outputs can improve trade-off between achievable performance and stability robustness to model uncertainty. We used an ANC problem example to show the benefits of adding feedback from output z compared to the original SISO design. In practice, if both performance and stability objectives are stated as H_∞ minimization problems, it may be more convenient to synthesize C_z and C_y simultaneously. We have found in the simulations of this ANC example that the two methods yield comparable results. Nevertheless, in some situations the two-step design may be useful. For example, a specific weighting function can be constructed and used to shape the free parameter q for better closed-loop stability robustness. It is also under our investigation whether this synthesis scheme is useful in multiobjective control problems; i.e., an H_2/H_∞ design procedure may be decoupled to minimizing $||W_p(P_{zd} - P_{zu} r P_{yd})||_2$ and $||W_s(1 - qP_{zu})||_\infty$ separately.

Finally, since our analysis is based on coprime factorization theories from [11] that were primarily developed for MIMO systems, we believe that with some effort our concept could be extended to MIMO problems. Another possible future research is to apply the analysis in Section 4 to unstable plants.

References

[1] H.W. Bode, *Network Analysis and Feedback Amplifier Design,* D. Van Nostrand, New York, 1945.

[2] J.S. Freudenberg and D.P. Looze, "Right Half Plane Poles and Zeros and Design Tradeoffs in Feedback Systems," *IEEE Trans. Automatic Control*, Vol. AC-30(6) 1985.

[3] J. Hong and D.S. Bernstein, "Bode Integral Constraints, Collocation, and Spillover in Active Noise and Vibration Control," *IEEE Trans. Control Systems Technology*, Vol. 6(1), pp. 111-120, 1998.

[4] J.F. Freudenberg, R.H. Middleton and A. Stefanopoulou, "A Survey of Inherent Design Limitations," *Proceedings of the American Control Conference*, Chicago, IL, 2000.

[5] M.M. Seron, J.H. Braslavsky and G.C. Goodwin, *Fundamental Limitations in Filtering and Control,* Springer-Verlag, 1997.

[6] V. Toochinda, T. Klawitter, C.V. Hollot and Y.Chait, "A Single-Input Two-Output Feedback Formulation for ANC Problems," *Proceedings of the American Control Conference*, Arlington, VA, 2001.

[7] V. Toochinda, C.V. Hollot and Y. Chait,"On Selecting Sensor and Actuator Locations for ANC in Ducts," *IEEE Conference on Decision and Control.*, Orlando, FL, 2001.

[8] V. Toochinda, "Fundamental Limitations of ANC in One-Dimensional Ducts using 2 Sensors and 1 Actuator," Ph.D. Dissertation, University of Massachusetts, Amherst, 2001.

[9] V. Marcopoli, "Inherent Limitations Associated with the Weapon Pointing Control Problem for a Tank Elevation System," *Proceedings of the American Control Conference,* Chicago, IL, 2000.

[10] I.M. Horowitz, *Syntheis of Feedback Systems,* Academic Press, 1963.

[11] M. Vidyasagar, *Control System Synthesis: A Factorization Approach,* MIT Press, Cambridge, MA, 1985.

[12] J.C. Doyle, B.A. Francis and A.R. Tannenbaum, *Feedback Control Theory,* Macmillan, New York, 1992.

[13] R.Y. Chiang and M.G. Safonov, *Robust Control Toolbox*, The MathWorks, Natick, MA, 1992.

[14] D. MacMartin and J. How, "Implementation and Prevention of Unstable Optimal Compensators," *Proceedings of the American Control Conference,* pp. 2190-2195, Baltimore, MD, 1994.