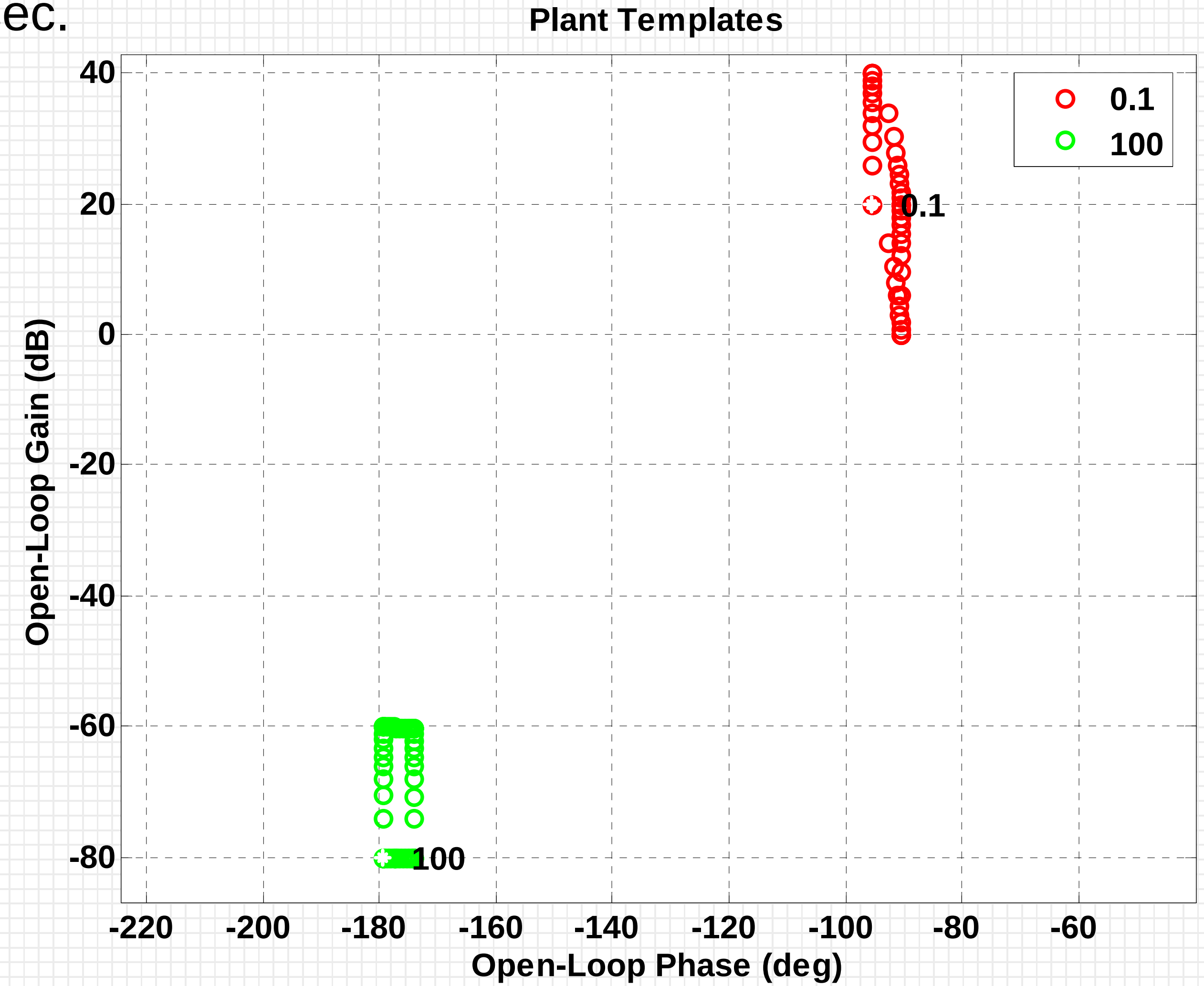
5.4. Additional Properties of Bounds and Templates

Clearly, $\mathcal{P}(\omega)$ is a nonlinear function of frequency. However, we can exploit its asymptotic behavior:

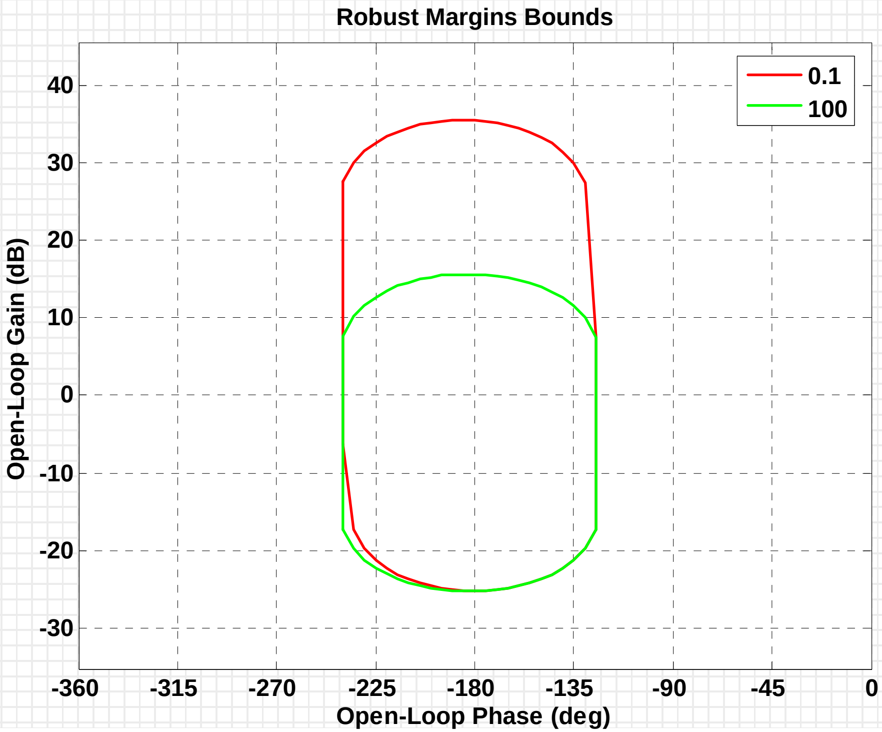
So

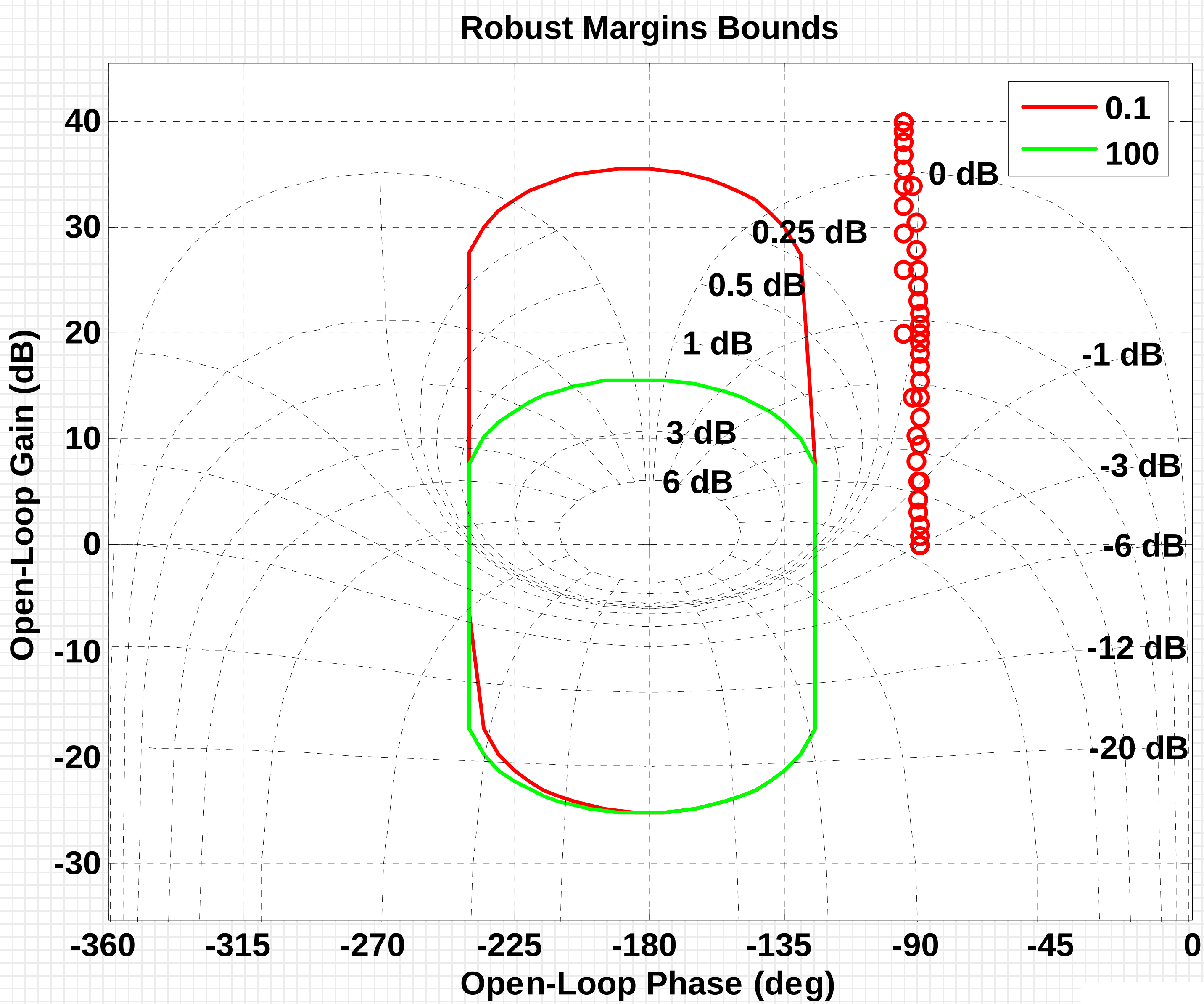
Now use `ch5_t.m` to re-compute the template at 0.1 and 100 rad/sec.



That is, the shape of these templates approaches a vertical line since the underlying uncertain plant exhibits only magnitude uncertainty.

Margin bounds (i.e., $|T| \leq \alpha|$) for such "skinny" templates resemble a vertically stretched M-circle. To see this, run `ch5_b.m/ch5_t.m` (nom is 21) A more detailed interpretation is shown in the next page.

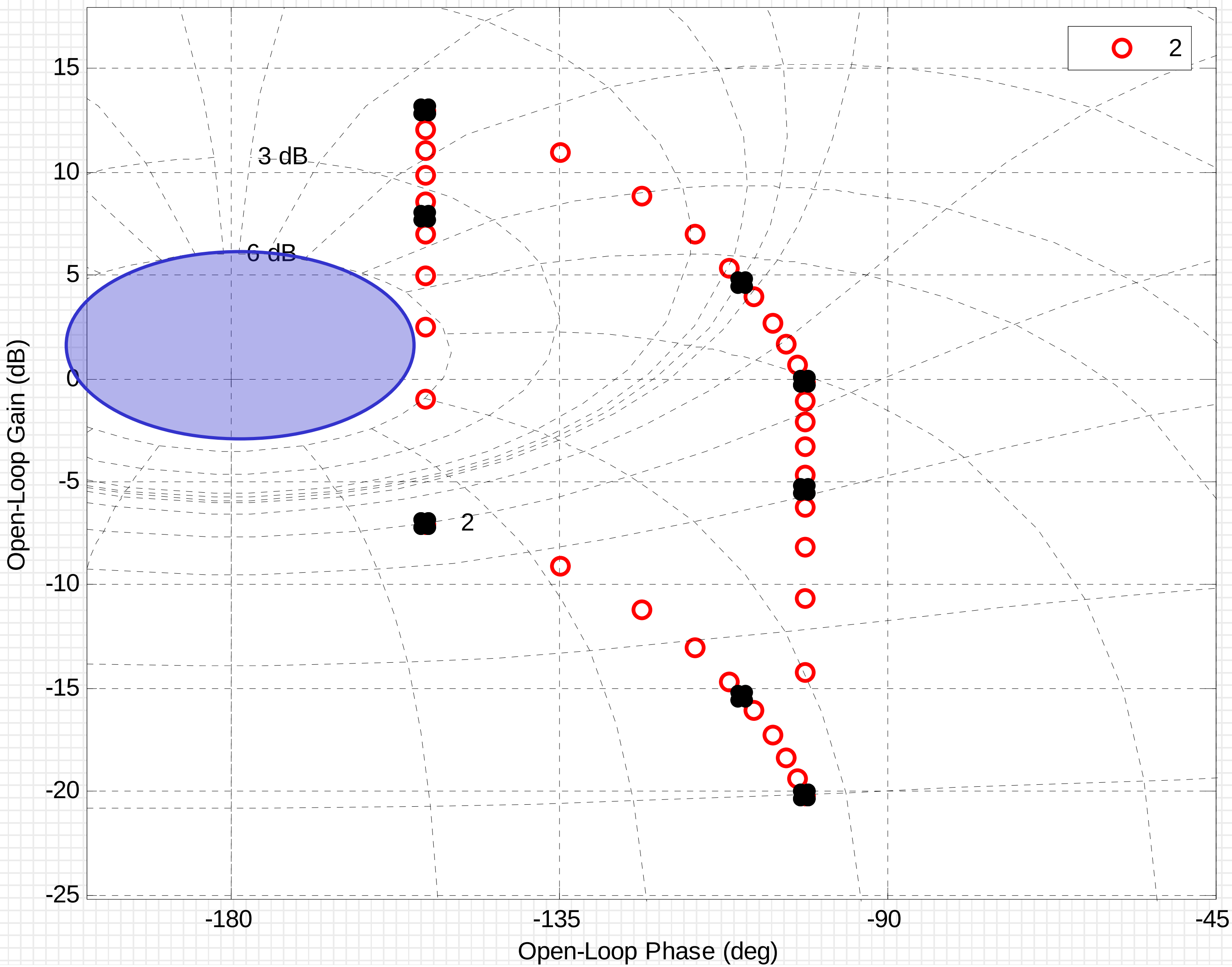



Robust Margins Bounds
0.1
100
0 dB
0.25 dB
0.5 dB
1 dB
3 dB
6 dB
-1 dB
-3 dB
-6 dB
-12 dB
-20 dB
40
30
20
10
0
-10
-20
-30
Open-Loop Gain (dB)
-360
-315
-270
-225
-180
-135
-90
-45
0
Open-Loop Phase (deg)

We conclude that:

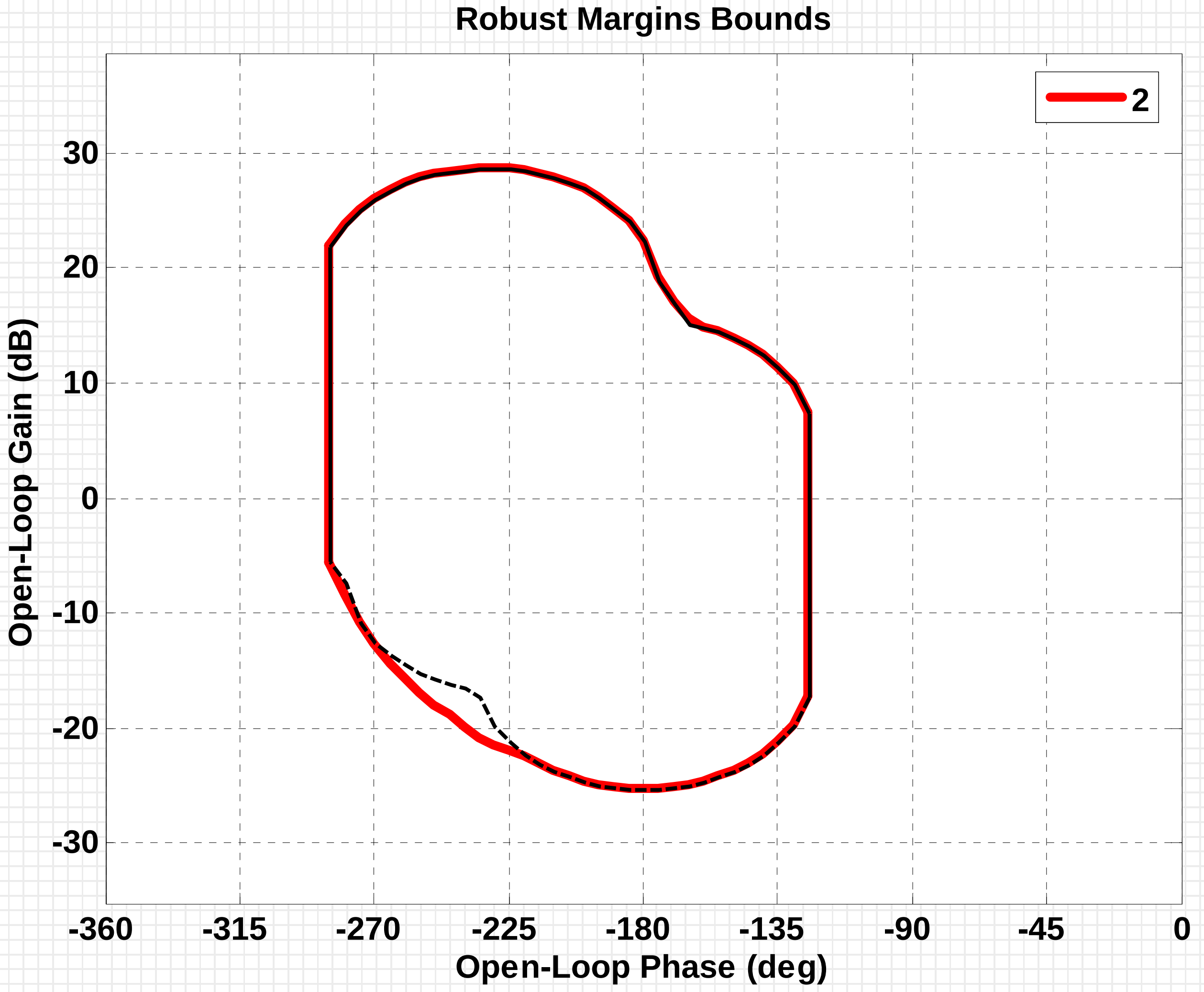
What is the effect of the accuracy of the template's boundary on the accuracy of the resulting bound? This is not a critical "problem" when computed manually (why?), but can be an issue with algorithms that only use discrete template points.

To see this, run `ch5_t.m` but modify the grid for each edge. Use 3 points instead of 10. Both templates are shown in the next page at $\omega = 2$.



2
3 dB
6 dB
2
15
10
5
0
-5
-10
-15
-20
-25
Open-Loop Gain (dB)
-180
-135
-90
-45
Open-Loop Phase (deg)

Both bounds are shown below at $\omega = 2$ for the nominal plant $(a_0,k_0)=(1,1)$.



What classes of uncertain plants result in phase uncertainty at low and high frequencies?

What about "typical" templates at "mid"-range frequencies?

5.5. Theoretical Formulation of Bounds

We are given a plant family $P \in \mathcal{P}$ and a closed disk $\mathcal{T}$ centered at the origin with radius α. We are asked to find a fixed controller $C(s)$ such that the complimentary sensitivity function $\mathcal{T}$

$$T = \frac{CP}{1 + CP}$$

satisfies

$$|T(j\omega)| \in \mathcal{T}, \quad \text{for all } P \in \mathcal{P}.$$

Consider the bilinear mapping $f: z \to w$ (i.e., $CP \to T$)

and its inverse $g: w \to z$ ($T \to CP$)

Then

and our spec becomes: find *C* such that

If $0 \notin \mathcal{P}$, we can re-write the above as

or

Now introduce a nominal plant $P_0 \neq 0$ (for stability/analytic considerations)

Generally speaking, such bounds are not convex and may even be non-connected.

In QFT, we compute such bounds at a discrete set of frequencies and display only their boundaries (implicitly assuming they are simply connected).

To solve the problem, design a nominal loop $L_0=CP_0$ satisfying the above QFT bounds and robustly stabilizes the plant.

5.6. Numerical Algorithms

Consider the spec on the complimentary sensitivity function T

$$|T(j\omega)| = \left|\frac{CP}{1+CP}(j\omega)\right| \le \alpha(\omega), \quad \forall P \in \{P_1, P_2, \ldots, P_n\}.$$

Note that we already assume that the plant set has been discretized in some way.

Let

$$C = ce^{j\phi} \quad \text{and} \quad P = pe^{j\theta}$$

and plug into the inequality above

Evaluate magnitude, then square both sides

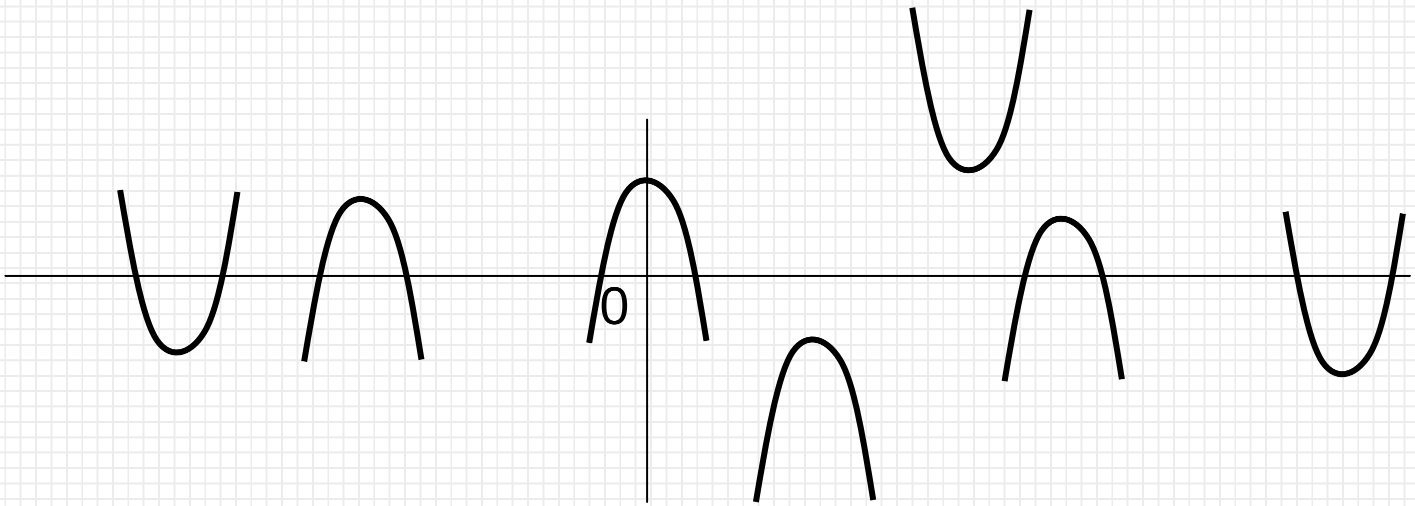
Invert and re-arrange to form a quadratic inequality

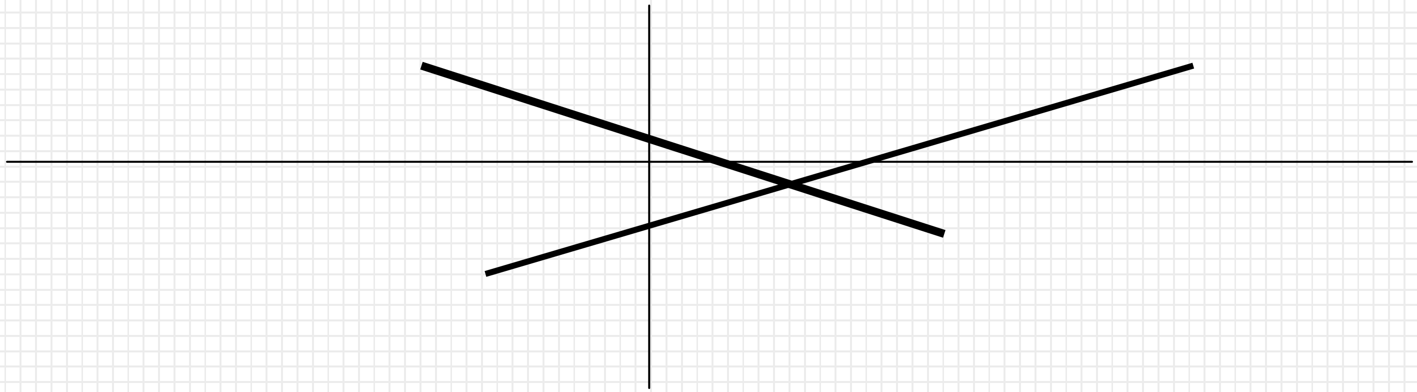
In fact, any of the performance specs (ptype) leads to a standard quadratic inequality of the form:

$$a(\mathrm{p},\phi,\theta,\alpha)c^2 + 2b(\mathrm{p},\phi,\theta,\alpha)c + \mathrm{d}(\mathrm{p},\phi,\theta,\alpha) \geq 0.$$

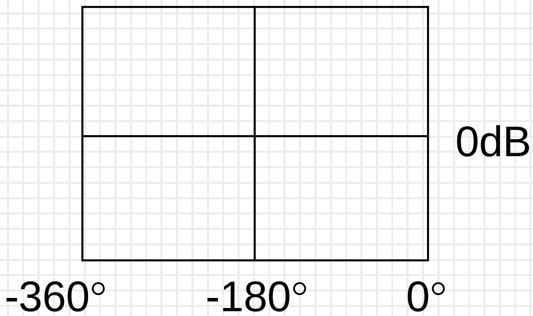
Graphically speaking, solution to the above can take on the following forms (assuming fixed controller phase ϕ):

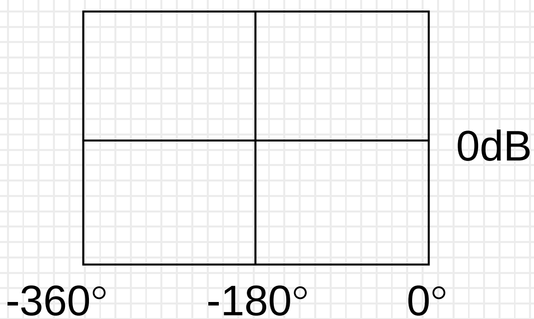
$$f(c) = ac^2 + 2bc + d \geq 0$$

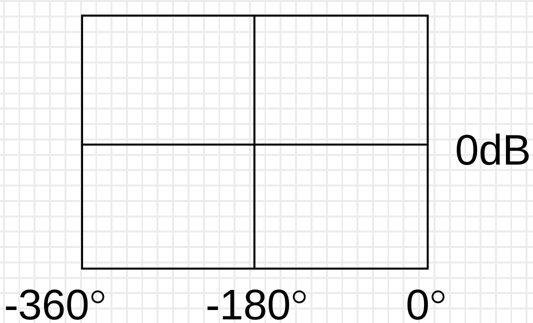




The three basic bounds are shown below.







A brute force algorithm (used in QFT toolbox) involves the following

- Fix the frequency
- Discretize controller phase $\phi \in (-360°, 0°]$ (toolbox default is 5°).
- Compute C_i for each plant P_i in the discrete set (done at each phase in the above range).
- Compute the intersection of all the individual bounds $C = \cap C_i$ (done at each phase in the above range).
- The bound on the nominal loop is simply P_0C where P_0 is any plant from the discretized set.

These algorithms are applicable to single-loop and multi-loop systems. More details can be found in

- Chait, Y., and Yaniv, O., “Multi-input/single-output computer-aided control design using the Quantitative Feedback Theory,” *Int. J. Robust and Nonlinear Control*, Vol. 3, pg. 47-54, 1993.
- Chait, Y., Borghesani, C., and Zheng, Y., “Single-loop QFT design for robust performance in the presence of non-parametric uncertainties,” *J. Dynamic Systems, Measurement, and Control*, Vol. 117, pp. 420-425, 1995.

5.7. HOMEWORK

1. Relate the spec on the sensitivity function

$$S = \frac{1}{1+L}$$

   to lower gain and phase margins.

2. Relate the spec on the complimentary sensitivity function to the upper gain and phase margins.