3. Plant Models

- Parametric uncertainty

$$\mathcal{P} = \left\{ P(s) = \frac{ka}{s(s+a)}: \quad k \in [1,10], \;\; a \in [1,10] \right\}$$

- Non-parametric uncertainty

$$\mathcal{P} = \left\{ P(s) = T_0(s)(1+\Delta_m(s)): \quad \left|\Delta_m(j\omega)\right| < R_m(\omega), \quad \Delta_m(s) \text{ stable} \right\}$$

- Discrete set

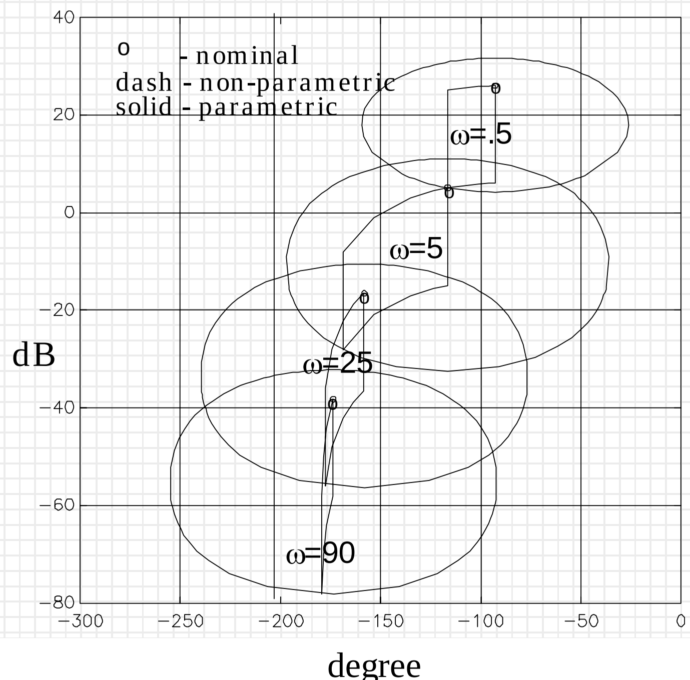
3.1. Plant Templates

Template ≡

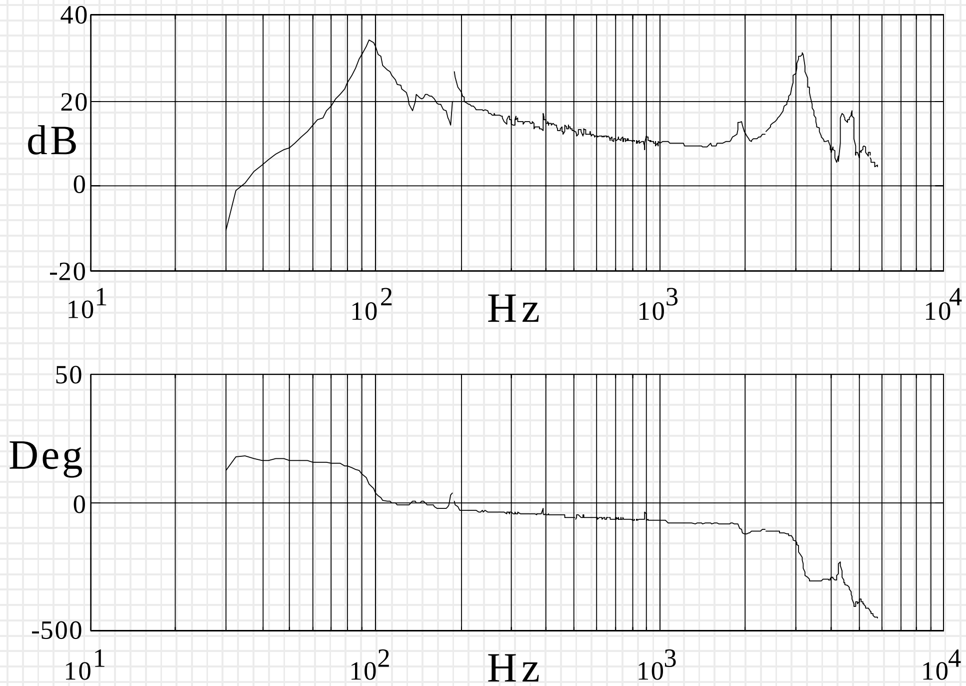
An example: parametric plant with its non-parametric representation

$$\mathcal{P}_1 = \left\{ P(s) = \frac{ka}{s(s+a)} : \quad k \in [1,10], \;\; a \in [1,10] \right\}$$

$$\mathcal{P}_2 = \left\{ \begin{array}{l} P(s) = P_0(s)(1+\Delta_m(s)) : \quad |\Delta_m(j\omega)| < R_m(\omega), \quad \Delta_m(s) \;\; \text{stable} \\ R_m(\omega) = \left| \frac{0.9(j\omega/0.91+1)}{(j\omega/1.001+1)} \right|, \quad P_0 = \frac{10 \cdot 10}{(s+10)} \end{array} \right\}$$



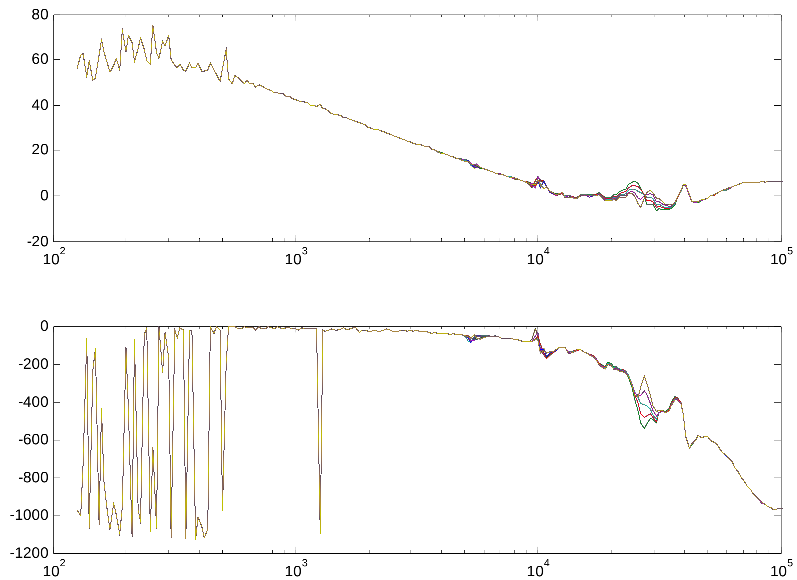
3.2.1. Measured (single) Response



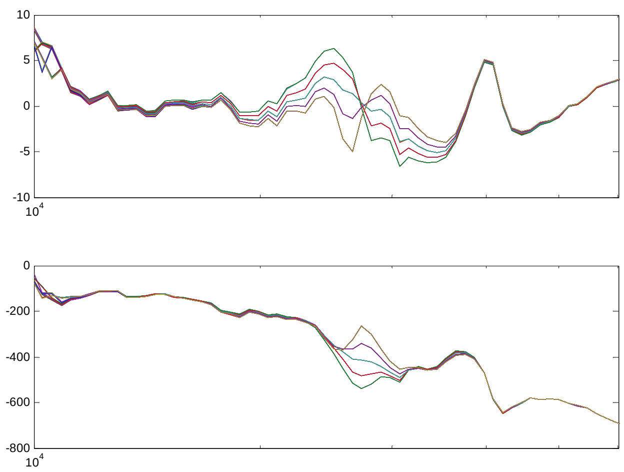
Single-axis active vibration isolation
(courtesy of LORD Corporation, Cary, NC).

The experimental plant frequency response is from an accelerometer mounted on a structure to an active mount connecting the structure to a vibrating engine.

3.2.2. Measured (Multiple) Responses



Measured nominal radial open-loop frequency response of a Compact Disc mechanism



3.3. Constructing Plant Templates

Consider a general description of an uncertain plant

$$P(s) = \frac{N(s,q)}{D(s,q)} = \frac{n_0(s) + \sum_{i=1}^{m} n_i(q) s^i}{d_0(s) + \sum_{j=1}^{n} n_j(q) s^j},$$

where $q \in Q \subset R^P$ is an uncertain parameter vector. The set Q is given by

$$Q = \left\{ q : \quad q \in \mathrm{R}^p, \underline{q}_i \le q_i \le \overline{q}^i, i = 1, \ldots, p \right\}.$$

At a fixed frequency, the set of responses of all possible plants is called plant template (in either complex plane or Nichols chart):

3.3.1 Constructing a Template's Boundary

- The most common way of generating templates is to grid the parameter space. If we're after the maximum mag of a closed-loop relation, we need consider only $\partial\mathcal{P}(\omega)$ (under some conditions and the Maximum Principle). Hence, deliberate gridding can avoid interior template points that just add computational load.

- Analytic results for efficient generation of templates are available. They use concepts such as Kharitonov polynomials, zeros of sets and Interval math. Related M-files are available. For example (see references therein):

  - Wen-Hua Chen and Donald J. Ballance. "Plant template generation of uncertain plants in Quantitative Feedback Theory." *ASME Journal of Dynamic Systems, Measurement and Control*, 121(3):358-364, 1999. (http://www.mech.gla.ac.uk/~donald/)

  - P. S. V. Nataraj and S. Sheela. "A Template generation algorithm using vectorized function evaluations and adaptive subdivisions." P. S. V. Nataraj and S. Sheela, Trans. *ASME J. Dynamical Systems, Measurement, and Control*, to appear. (http://www.ee.iitb.ac.in/~nataraj/publications.htm)

3.4. Uncertainty Description in the Toolbox

- Parametric uncertainty is approximated (discretized) both in shape and frequency.

- Non-parametric uncertainty is defined by a nominal response and the radius for the multiplicative uncertainty (at discrete set of frequencies).

$$P_0(s) \quad \text{and} \quad \Delta_m(s)$$

3.4.1 Templates in the Toolbox: Arrays of LTI Models

Consider a parametric plant, e.g.,

$$\mathcal{P} = \left\{ P(s) = \frac{k}{(s+a)(s+b)} : \quad k \in [1,10], \;\; a \in [1,5], \;\; b \in [20,30] \right\}$$

The QFT Toolbox relies on Control Toolbox's LTI models. In this context:

```
c = 1; k = 10; b = 20;
for a = logspace(log10(1),log10(5),10),
  P(1,1,c) = tf(k,[1,a+b,a*b]);
  c = c + 1;
end
```

```
>> size(P)
10x1 array of transfer functions
Each model has 1 output and 1 input.
```


3.4.2. Templates in the Toolbox: Frequency Response Sets

Models given by a frequency response set are defined using Control Toolbox's arrays of FRD models :

```
w = logspace(-1,1,100);
Pw = freqresp(P,w);
```

```
>> size(Pw)
ans =
     1     1   100    10
```

`freqresp` outputs a 4-dim matrix – not an FRD model. Also, it is often necessary to eliminate the singleton dimensions

```
>> Pw = squeeze(Pw);
>> size(Pw)
ans =
   100    10
```

Since math operations are simpler with LTI objects, we convert Pw into an FRD array

```
>> Pw = frd(Pw,w);
>> size(Pw)
10x1 array of FRD models.
Each model has 1 output and 1 input, at
100 frequency points.
```

In summary, the QFT Toolbox (V2.x) relies on LTI/FRD models. There are many advantages and few disadvantages. The chief advantage is that it allows for algebraic manipulations with mixed models and arrays (as we shall see later).

3.4.3. Templates in the Toolbox: Related Functions

Operation	
`addtmpl`	Add LTI/FRD arrays
`cltmpl`	Closed-loop LTI/FRD arrays from open-loop arrays
`multmpl`	Multiply LTI/FRD arrays

For example, consider addition of two transfer function sets given by

$$P_1(s) = \frac{1}{s+a}, \quad a \in [1,10] \qquad P_2(s) = \frac{b}{s+2}, \quad b \in [0.1,0.5]$$

First form LTI arrays to represent the above models using linear parameter space grids

```
c = 1;
for a = linspace(1,10,10),
 P1(1,1,c) = tf(1,[1,a]);  c = c + 1;
end

c = 1;
for b = linspace(0.1,0.5,10),
 P2(1,1,c) = tf(b,[1,2]);  c = c + 1;
end
```

The addition is computed using

```
P = addtmpl(P1,P2,2);
```

```
>> size(P1)
10x1 array of transfer functions
Each model has 1 output and 1 input

>> size(P2)
10x1 array of transfer functions
Each model has 1 output and 1 input.

>> size(P)
100x1 array of transfer functions
Each model has 1 output and 1 input.
```

Note that using either

```
P = P1+P2;  % objects!
```

or

```
P = addtmpl(P1,P2);  % QFT function!
```

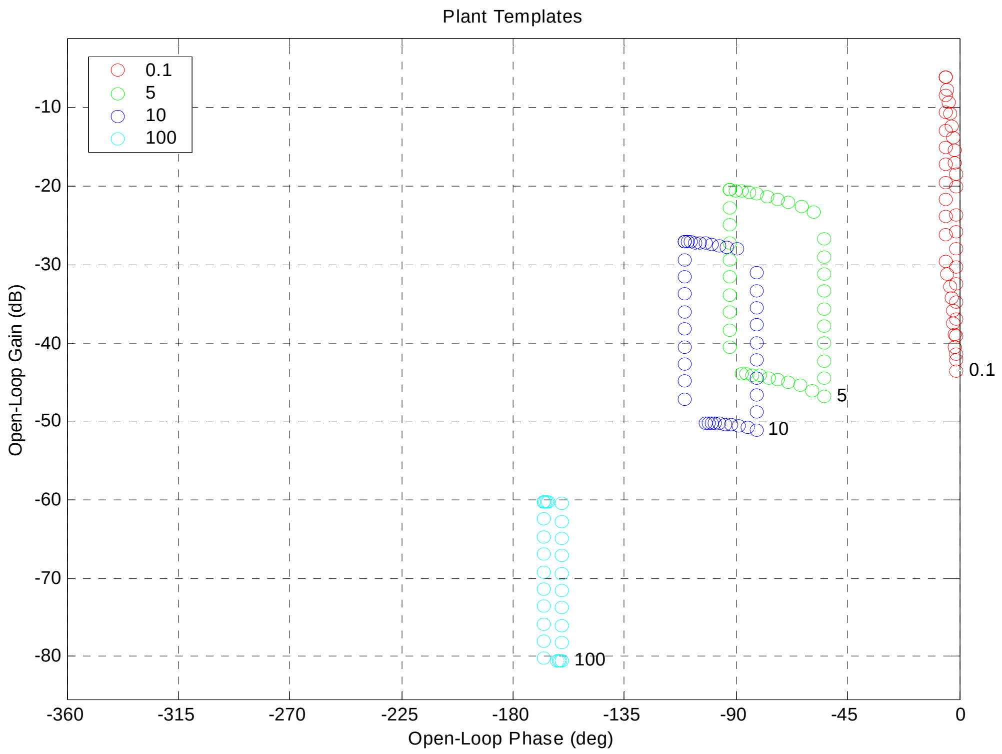
results in an erroneous addition since both assume correlated uncertainties (thought the results are different in that different elements are paired in the summation)

```
>> size(P1+P2)
10x1 array of transfer functions
Each model has 1 output and 1 input.
```

3.5. Template Plotting

To view computed templates:

```
plottmpl(w,P,nom)
```



The code for mapping boundaries only is:

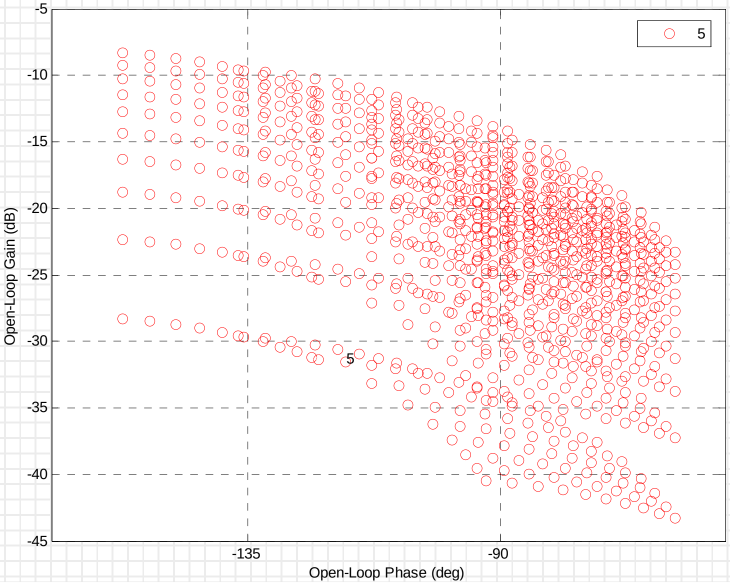
```
c = 1; k = 10; b = 20;
for a = linspace(1,5,10),
 P(1,1,c) = tf(k,[1,a+b,a*b]);  c = c + 1;
end
k = 1; b = 30;
for a = linspace(1,5,10),
 P(1,1,c) = tf(k,[1,a+b,a*b]);  c = c + 1;
end
b = 30; a = 5;
for k = linspace(1,10,10),
 P(1,1,c) = tf(k, [1,a+b,a*b]);  c = c + 1;
end
b = 20; a = 1;
for k = linspace(1,10,10),
 P(1,1,c) = tf(k, [1,a+b,a*b]);  c = c + 1;
end
nompt = 21;  % define nominal plant case
```

```
>> size(P)
40x1 array of transfer functions
Each model has 1 output and 1 input.
```

While a crude gridding looks like:

```
c = 1;
for a = linspace(1,5,10),
 for b = linspace(1,20,10),
  for k = linspace(1,10,10),
   P(1,1,c) = tf(k, [1,a+b,a*b]);  c = c + 1;
  end
 end
end
nompt = 21;  % define nominal plant case
```

```
>> size(P)
1000x1 array of transfer functions
Each model has 1 output and 1
input.
```



3.6. Other Considerations

3.7. Homework

Compute and plot the following plant templates with minimal interior points (apply analytical results).

1. At $\omega = 3$:

$$P(s) = \frac{0.7}{(s+a)(s+b)}, \quad \mathcal{Q} = \{(a,b): a \in [1,3], b \in [1,14]\}.$$

2. At $\omega = 7$:

$$P(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\mathcal{Q} = \{(\zeta, \omega_n): \ \zeta \in [0.01, 0.05], \omega_n \in [5,10]\}.$$

3. At $\omega = 1$:

$$P(s) = \frac{(s+a)}{s(s+b)(s+c)}, \quad \mathcal{Q} = \{(a,b,c): a \in [1,4], b \in [1,14], c \in [1,6]\}.$$