17. OPTIMAL CONTROL

In this chapter we briefly discuss a widely used method for synthesizing controllers. We start with Linear Quadratic Regulator (LQR) which uses all the plant's states for feedback

We then present an overview of Linear Quadratic Gaussian (LQG) for problems where not all states can be measured and there exists process and sensor noise. LQG computes the optimal observer needed to estimate the plan's states (which are needed for LQR).

We conclude by presenting Loop Transfer Recovery (LTR) which is aimed at improving stability margins.

17.1. Preliminaries. A system model can be written in the MTF format, or in state space form which shows the differential equations underlying the MTF. For example, consider a siso plant

$$P = \frac{y}{u} = \frac{2}{s^2 + 7s + 12}.$$

Assuming zero initial conditions, the linear differential equation that gave rise to the above is

$$\ddot{y}(t) + 7\dot{y}(t) + 12\mathrm{y}(t) = 2\mathrm{u}(t).$$

We assign *states* to the model, one for each derivative (2 here). There are several ways to do this. For example, define

$$x_1 = y$$
$$x_2 = \dot{y}$$

so

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{y} = 2\mathrm{u}(t) - 7\dot{y}(t) - 12\mathrm{y}(t) \\ &= 2\mathrm{u}(t) - 7x_2(t) - 12x_1(t)\end{aligned}$$

This state-space model has the matrix form

$$\begin{bmatrix}\dot{x}_1(t) \\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -12 & -7\end{bmatrix}\begin{bmatrix}x_1(t) \\ x_2(t)\end{bmatrix} + \begin{bmatrix}0 \\ 2\end{bmatrix}u(t).$$

The system output $y(t)$ is computed from the states

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

The general state-space form is

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where D denotes a direct connection from the inputs to the outputs (i.e., proper MTFs).

Note that there a number of ways we can represent the model in state-space form, all similar to each other via similarity transformations. Also

$$P(s) = C(sI - A)^{-1}B + D.$$

As discussed in Chap. 13, the poles of the MTF appear in the eigenvalues of *A*. The zeros of an MTF are computed using the state-space form.

Some related MATLAB commands are

```
sys = ss(A,B,C,D)

sys = tf(N,Den)

[A,B,C,D] = ssdata(sys,’v’)

tzero(sys)

eig(sys)  or eig(A)
```

It is possible that given the state-space structure, some of the states cannot be affected by control action. In SISO, this means pole/zero cancellation; this also occurs in MIMO systems, but may not be seen directly from the MTF. The term *controllability* describes this property.

Similarly, there are states not seen in the output due to pole/zero cancellation. The term *observability* describes this property.

In what follows we assume that the system is controllable and observable (do not assume that as a rule). The MATALB functions to check these are `CTRB` and `OBSV`.

17. Other Techniques

17.1. Linear Quadratic Regulator. LQR is an optimization problem that synthesizes a full state-feedback law

$$u = -K_c x$$

that minimizes a quadratic performance index of the form

$$J = \int_0^\infty \left(x^T Q x + u^T R u \right) dt$$

where Q and R are weighting matrices satisfying

Q is positive semidefinite: $Q = Q^T \geq 0$

R is positive definite: $R = R^T > 0$

The control law that minimizes the cost function is given by

$$K_c = R^{-1} B^T P$$

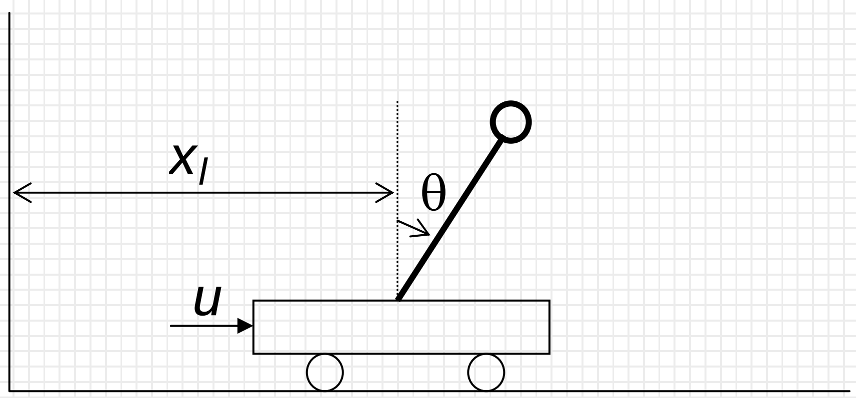
where P is the solution of the Algebraic Riccati Equation (ARE)

$$0 = PA + A^T P + Q - PBR^{-1}B^T P.$$

If the unstable modes of the system are both observable and controllable, then there exists such a P.

The selection of Q and R to generate desired closed-loop system dynamics is iterative. Nevertheless, the insight gain during such iterations at the initial design phase is exploited at the latter stages. Next, let us consider an illustrative example (`ltr_ex.m`).

The model for an inverted pendulum (see *Sidi* for details)



linearized about the "up" position is

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -4.905 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 29.43 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} u, \qquad x = \begin{bmatrix} x_l \\ \dot{x}_l \\ \theta \\ \dot{\theta} \end{bmatrix}.$$

The plant has 2 integrators and one unstable pole (why?).

```
>> eig(A)
ans =
         0
         0
    5.4249
   -5.4249
```

Checking for controllability

```
>> rank(ctrb(A,B))
ans =
     4
```

Since the rank equals the number of states, this system is controllable.

Let us arbitrarily select

$$Q = \text{diag}[20,2,20,2], \quad R = 10.$$

Hence, the 1st and 3rd states are considered more "important" in terms of the cost function.

The solution can be obtained from either `LQR` or `CARE` functions. For example,

```
[P,EK,Kc] = care(A,B,Q,R);
```

The optimal state-feedback controller is $u = K_c x$

$$K_c = [\text{-1.4142} \quad \text{-2.6505} \quad \text{-38.3871} \quad \text{-7.4840}].$$

The closed-loop system matrix is

$$\dot{x} = Ax + Bu = Ax + BK_c x = (A + BK_c)x = A_{cl}x$$

whose eigenvalues are

```
(-15.5234,-2.2843+1.4678i,-2.2843-1.4678i,-3.4287).
```

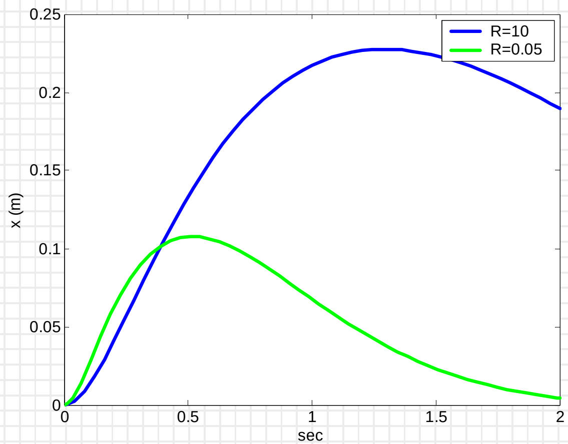
Another choice of weights with a relaxed control constraint

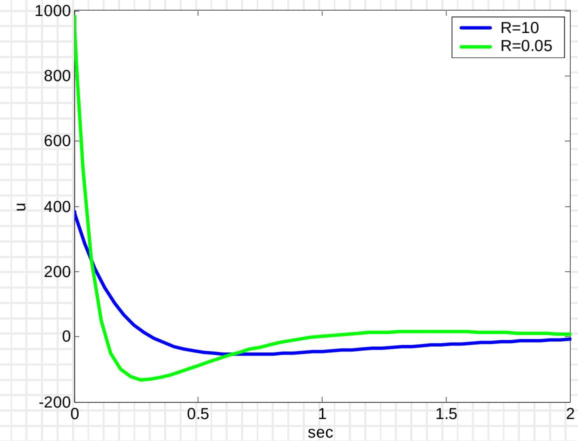
$$Q = \text{diag}[20,2,20,2], \;\; \text{R} = 0.05$$

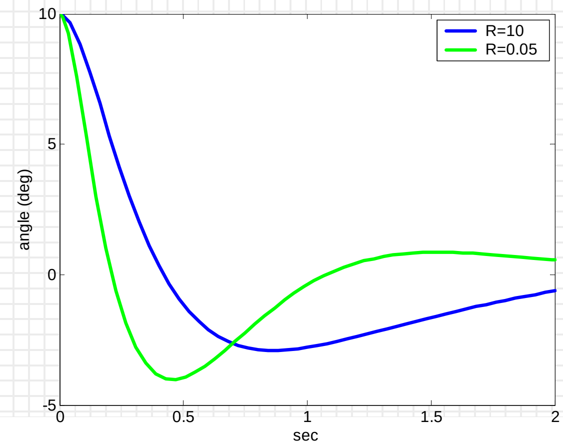
results in

$$K_c = [\text{-20.0000} \;\; \text{-19.5151} \;\; \text{-98.3058} \;\; \text{-21.5179}]$$

Responses for an initial condition of θ=10° is shown below.



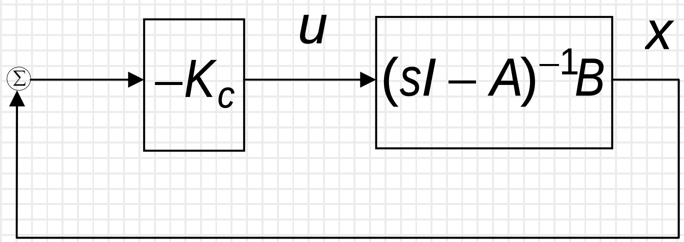




We observe that relaxing the weight on the control effort results in a larger $u(t)$ which in turn limits the maximal deviation of $x(t)$.

In general, Q and R are used to trade off performance of the states and control effort. There is no exact relation, but one can find numerous papers/books discussing this topic. A solution exists for any admissible pair (Q,R) (no RHP zeros!).

The margins computed for a loop transmission between a state and an input can be shown to be at least 60° PM and 2 (6 dB) GM. These loops always have a relative degree of one. Both of these properties are largely non-practical. The loop (with the system opened at the plant input is



$$L = K_c(sI - A)^{-1}B$$

it can be shown that (SISO)

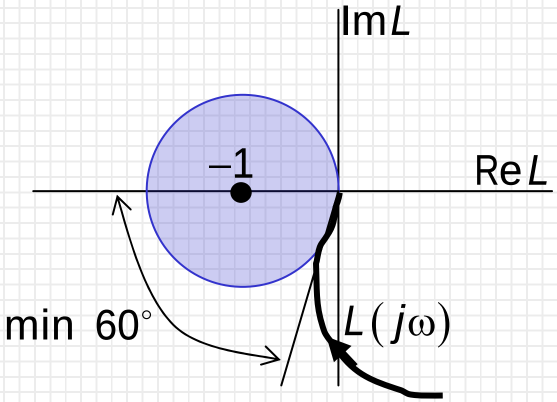
$$\angle L(j\omega) \xrightarrow[\omega\to\infty]{} -90^\circ$$

meaning the loop has one more pole than zero.

In addition, it can be shown that

$$\det(I+L)=\det\left(I+K_c\left(j\omega I-A\right)^{-1}B\right)\geq 1$$

which has this graphical interpretation



Similar relations were developed for a MIMO system. Specifically, the 60° PM and 2 (6 dB) GM properties hold for each p_{ii}^{n}. Of course, we recall that this alone does not guarantee MIMO margins!

Finally, the above LQR formulation has no inputs. It is possible to modify the setup to, for example, allow for a model-following problem.

17.2. Linear Quadratic Gaussian (LQG)

LQR requires knowledge of all the states. However, in practice, the only some of the states are measured (outputs). Consider a linear system whose dynamics are affected by a disturbance and the measurements are corrupted by noise

$$\dot{x} = Ax + Bu + w$$

$$y = Cx + v$$

where v and w are stationary, zero mean, Gaussian white noise processes with covariance matrices $V \geq 0$ and $W > 0$, respectively. It is assumed that v and w are uncorrelated, that is $E(wv^T) = 0$.

Next, consider the linear system

$$\dot{\hat{x}} = A\hat{x} + Bu + K_f(y - C\hat{x}).$$

where

$$K_f = \Sigma C^T V^{-1}$$

and where Σ is the solution to the ARE

$$0 = \Sigma A + \Sigma A^T + W - \Sigma C^T V^{-1} C\Sigma.$$

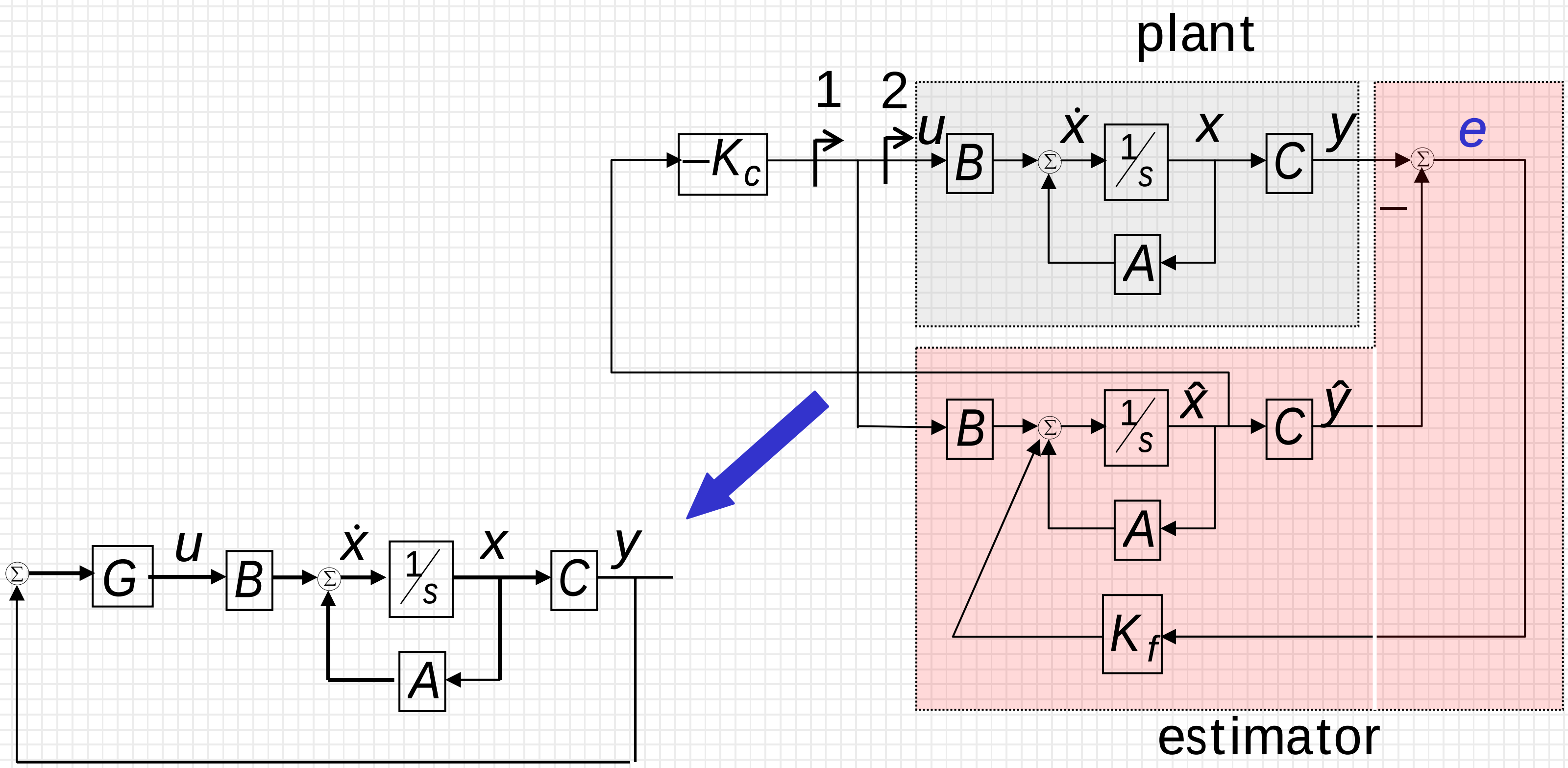
Then

$$\hat{x} \text{ minimizes } E\left((x-\hat{x})^T (x-\hat{x})\right)$$

and

$$E(x-\hat{x}) = 0.$$

The combined system is shown below.



This estimator is known as the Kalman-Bucy filter. A very famous invention with wide-ranging applications.

The full-state feedback law can be combined with the estimator to result in an dynamic, output feedback controller of the form

$$G = -K_c(sI - A - BK_c - K_f C)K_f .$$

The noise parameters V and W are used as design "knobs" to affect the dynamics of the overall controller G.

For example, assume that the linear velocity and the angular rates of the pendulum system are un-measurable. So

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x = Cx$$

$$v = [v_1, v_2]^T$$

$$w = [w_1, w_2 w_3, w_4]^T$$

and let

$$W = \text{diag}(0.02^2, 5^2, 0.01^2, 5^2), \;\; V = \text{diag}(0.01^2, 0.01^2).$$

The solution can be obtained from either LQG or CARE functions. For example,

```
[Sigma,EKf,Kf] = care(A',C',W,V);
```

The optimal filter gain is

$$K_f = \begin{bmatrix} 31.69 & -0.081 \\ 500.1 & -5.05 \\ -0.081 & 32.58 \\ -0.145 & 530.4 \end{bmatrix}.$$

The *separation principle* shows that the solution to the LQG problem involves separate designs of full-state LQR system and a Kalman-Bucy filter. Hence, a solution is guaranteed. The matrices Q, R, W, and V are used for tuning purposes.

17.2. Loop Transfer Recovery (LTR)

Doyle and Stein (1979) discovered a basic weakness of this design in terms of stability margins. To correctly analyze robustness vs. plant uncertainties, the loop must be opened at point 2 which is the plant input *u*. When the loop is opened at point 1, the same uncertainty will be seen also by the estimator which effectively nullifies the fact the estimator design is based on a nominal plant.

Loop Transfer Recovery (LTR) has been proposed to overcome this weakness. When the loop is broken at point 2, the loop is

$$L_2 = GC(sI - A)^{-1}B$$

$$= K_c(sI - A - BK_c - K_f C)K_f C(sI - A)^{-1}B$$

while when the loop is opened at point 1 we have

$$L_1 = K_c(sI - A - BK_c)^{-1}B$$

which is the original LQR loop.

LTR, by proper selection of W and V, attempts to recover the margins for a full-state feedback system, that is

$$L_2 \rightarrow L_1.$$

To do so, it can be shown that the covariance matrix W is to be augmented as follows

$$W_{LTR} = W + \rho^2 BSB^T$$

for some matrix $S \geq 0$ matrix. When $\rho \rightarrow \infty$, if P is MP, we can show that LTR's margins are fully recovered at loop point 2.

Essentially, $G(s)$ inverts the plant and adds some far-off poles. When the plant is NMP, this is not feasible. Nevertheless, we are not interested in having L_2 match L_1 at all frequencies since L_1 has a relative degree of 1 (hence, is sensitivity to noise at high frequencies).

It is possible to also recover loop properties at the plant output using a similar procedure.

LTR works best for a square, MP plant. If the plant is NMP, the usual limitation of such zeros is applicable (limited cross-over frequency) and recovery is available only up to a limited bandwidth (the LTR controller will not have the NMP zeros as its poles).

A non-square plant requires some additional work to square it out.

Example (Franklin et. al., 2002, undergrad control text). Consider a satellite attitude control problem with a state-space description (a double integrator) (`ltr_ex2.m`)

$$P = \frac{1}{s^2} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0.$$

We first design an LQR controller for

$$Q = C^T C, \quad R = 1.$$

The LQR's loop transfer function is

$$K_c (sI - A)^{-1} B = \frac{1.414(s + 0.707)}{s^2}.$$

Next, we design the LQG controller to recover loop response at the plant input using

$$\Gamma = q^2 B, \quad W = \Gamma\Gamma^T, \quad V = 1.$$

The compensator is (for q = 10)

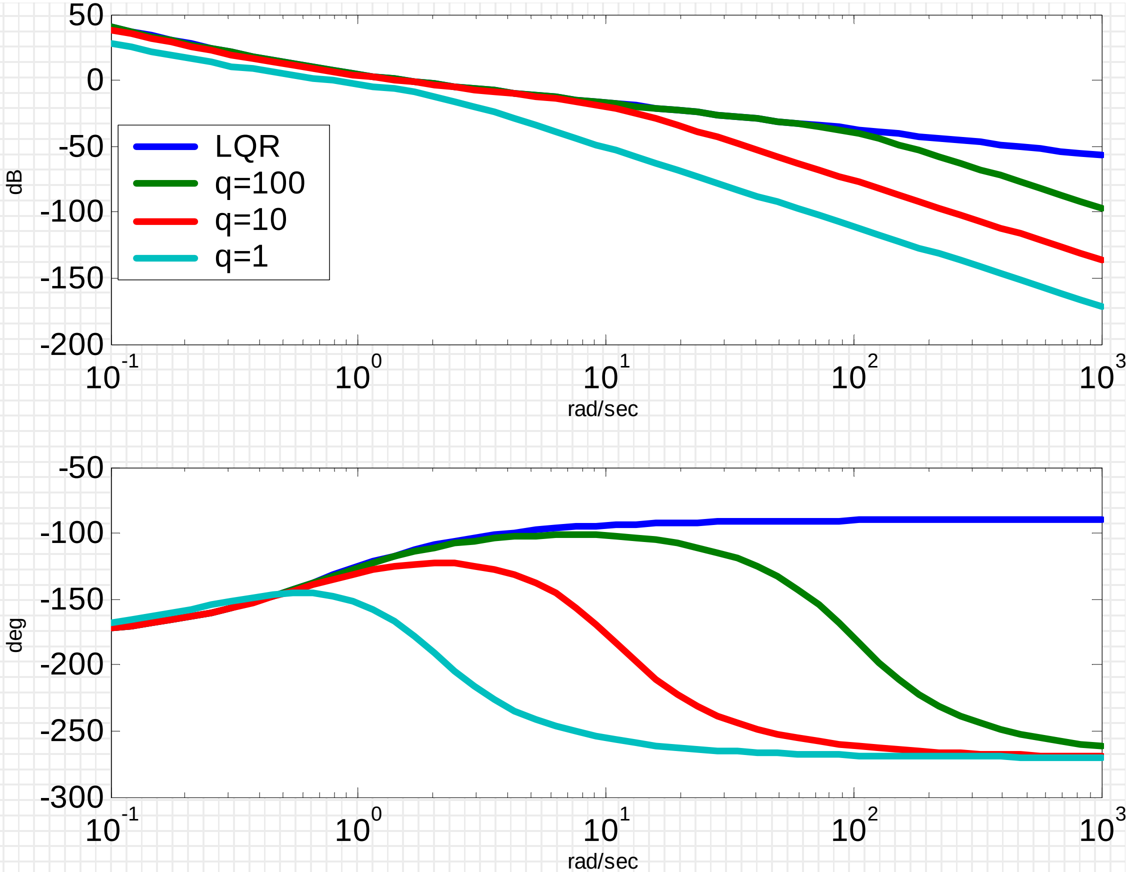
$$G = K_c(sI - A - BK_c - K_f C)^{-1} K_f = \frac{155.56(s + 0.6428)}{(s + 7.77 + j7.77)(s + 7.77 - j7.77)}.$$

And together with the double integrator plant it forms the loop transfer function

$$L = PG = \frac{155.56(s + 0.6428)}{s^2(s + 7.77 + j7.77)(s + 7.77 - j7.77)}$$

The figures next page compare the loops corresponding to LQR and LTR with q = 1, 10, and 100.

We observe that as *q* is increased, the LTR loop gain approaches that of the LTR, but not the phase. Note that the LQR's loop has a relative degree of 1, the same as the LTR's controller. Hence, the LTR loops has a total relative degree of 3 (2 from the plant).



LTR's margins were nearly recovered with $q = 10$: GM=2.1 dB and PM = 55°.

LQR, LQG, and LTR cannot explicitly account for plant uncertainty.

17.3. Stability Robustness with Unstructured Uncertainty

Recalling our uncertainty representations, assume the siso plant has a has a (input multiplicative) form

$$P(s) = P_0(s)(1 + W(s)\Delta(s)), \quad W, \Delta \text{ stable}, \quad \|\Delta(j\omega)\|_\infty < 1$$

where $W(s)$ is a weighting function used to specific a frequency-dependent level of uncertainty.

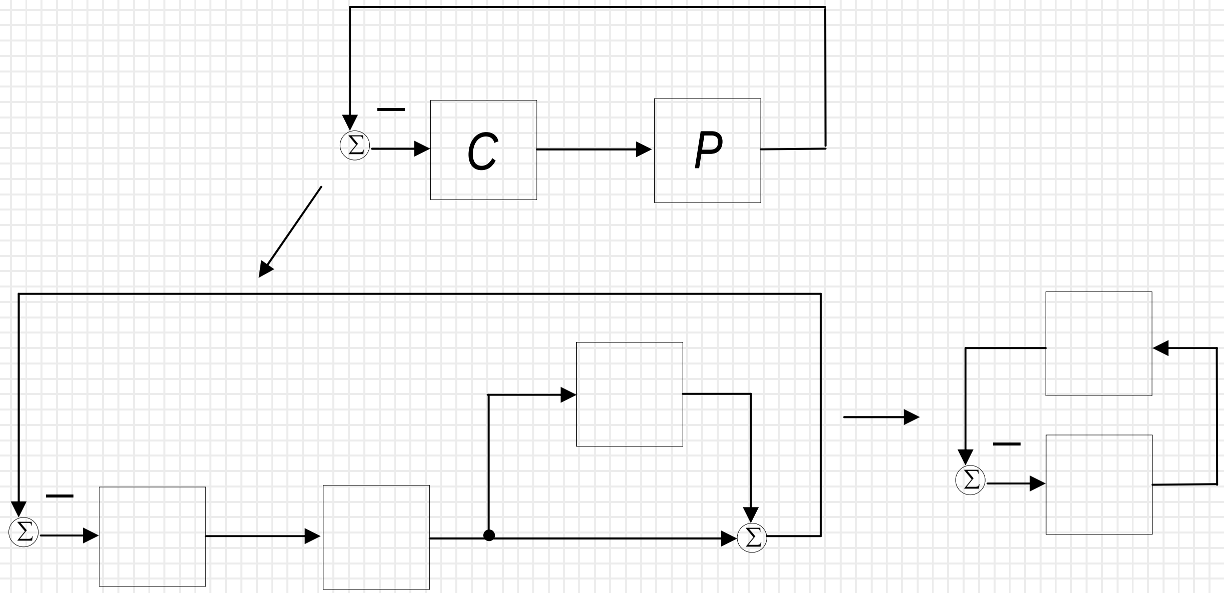
In addition we assume that the nominal closed-loop system is stable. Clearly, this system is stable if the Nyquist plot of CP has the same number of encirclements. Details of the proof are left out, but essentially, we have robust stability iff the Nyquist envelop does not include the (-1,0) point, namely

$$|1 + CP(j\omega)| > 0 \;\Leftrightarrow\; |1 + CP_0 + CP_0 W\Delta| > 0 \Leftrightarrow |1 + CP_0| \left|1 + \frac{CP_0}{1 + CP_0} W\Delta\right|$$

$$\Leftrightarrow \left|1 + \frac{CP_0}{1 + CP_0} W\Delta\right| > 0 \text{ } (1 + CP_0 \text{ has only stable roots}) \quad \Leftrightarrow |T_0 W\Delta| < 1$$

$$\Leftrightarrow \quad |T_0(j\omega)| < \frac{1}{|W(j\omega)|} \quad (\text{since } \|\Delta(j\omega)\|_\infty < 1)$$

This inequality can also be derived using block diagram algebra



Application of the small gain theorem says that we remain stable if

$$|T_0 W \Delta| < 1.$$

Another way to look at this uncertainty is from a performance stand point. Assume that plant is fixed $P = P_0$, and that in addition to stability we have a performance weight on the complimentary sensitivity function

$$|T(j\omega)| < \frac{1}{|W(j\omega)|} \quad .$$

Different uncertainty structures leads to constraints on different closed-loop functions. For example, with an additive uncertainty model

$$P(s) = P_0(s) + W(s)\Delta(s), \quad W,\Delta \text{ stable}, \quad \|\Delta(j\omega)\|_\infty < 1$$

We have robust stability iff

$$|CS_0(j\omega)| < \frac{1}{|W(j\omega)|} \quad .$$

In a mimo setting, an input multiplicative uncertainty has the form

$$P(s) = P_0(s)(1 + W_1(s)\Delta(s)W_2(s)),\ W_1W_2,\Delta \text{ stable}, \quad \|\Delta(j\omega)\|_\infty < 1$$

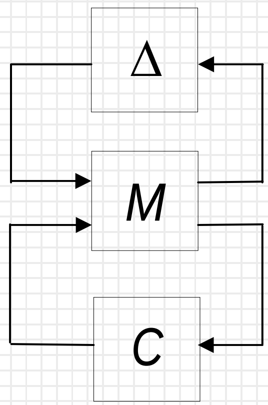
The weighting matrices are used to model the frequency and directional dependency of the uncertainty (since matrices do not commute in general). It can be shown that a necessary and sufficient condition for robust stability is

$$\|W_2 T_I W_1\|_\infty < 1.$$

For more details, there are numerous books on this topic. One excellent reference for feedback control see:

J.S. Freudenberg, C.V. Hollot and Looze, D.P., *A first graduate course in feedback control*, (contact hollot@ecs.umass.edu).

In a general mimo system with uncertainties we can re-draw the block diagram to have this form



where, for example, the Δ block is a 3x3 diagonal MTF

$$\Delta = \begin{bmatrix} \Delta_1 & 0 & 0 \\ 0 & \Delta_2 & 0 \\ 0 & 0 & \Delta_3 \end{bmatrix}$$

with some diagonal elements corresponding to plant uncertainty and some to performance weights.

The H_∞ control design method returns, if it exists, the optimal controller $C(s)$ such that it achieves robust stability and nominal performance with respect to fully populated, complex matrix Δ

$$\|\Delta\|_\infty < 1.$$

Hence, the name *unstructured* uncertainty. With some over design, H_∞ can achieve robust performance.

The μ-synthesis technique attempts to add pre and post scaling matrices to Δ to exploit its structure. In most cases, this is a nonlinear optimization.

