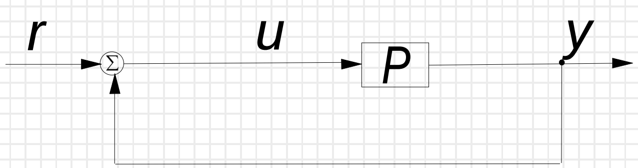
16. Multivariable QFT Design (Part 3)

16.1. Stability Margins. We motivate the study of MIMO stability margins problem using a satellite spinning about one of its axis (*J.C. Doyle*, 1986, and countless other refs.) The plant and block diagram are shown below (assume $C = I$, $a = 10$)



$$P = \frac{1}{s^2 + a^2}\begin{bmatrix} s - a^2 & a(s+1) \\ -a(s+1) & s - a^2 \end{bmatrix}$$

The complimentary sensitivity MTF is

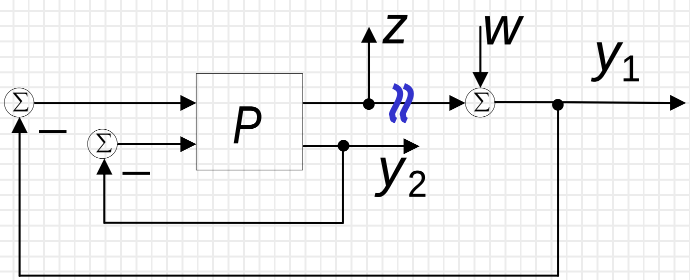
$$y = \frac{1}{s+1}\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} r = Tr$$

and the sensitivity MTF is

$$S = \frac{1}{s+1}\begin{bmatrix} s & a \\ -a & s \end{bmatrix}.$$

The closed-loop system is stable. But how sensitive is it to parameter variations? To answer this question, we should first quantify the uncertainty.

An apparent weakness in for inferring stability robustness based on the loops' margins can be shown as follows. If we open the 1st loop between *z* and *w* as shown below



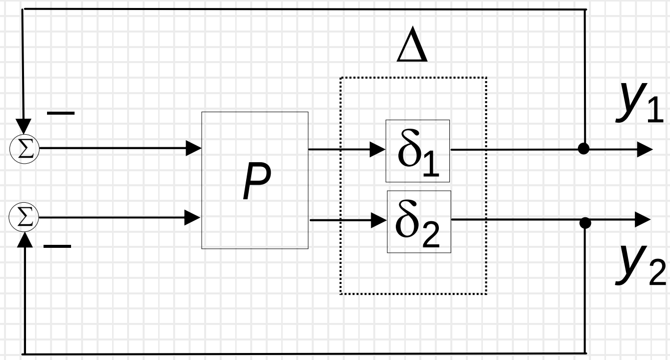
For simplicity we use the direct (i.e., non-inversion) procedure where (the 2nd loop was closed 1st)

This loop can tolerate an infinite gain uncertainty inserted between *z* and *w*, regardless of the value of *a*.

The same conclusion also holds if we study the open-loop MTF with the 1st loop closed and the 2nd loop opened in a similar location.

Since both loops have infinite gain margins, one might be led to conclude that the MIMO system also possesses such an excellent property. This is an incorrect conclusion as shown next.

Assume the both loops have gain uncertainty as shown below,



then, it can be shown that the closed-loop system is stable if the following function has stable roots

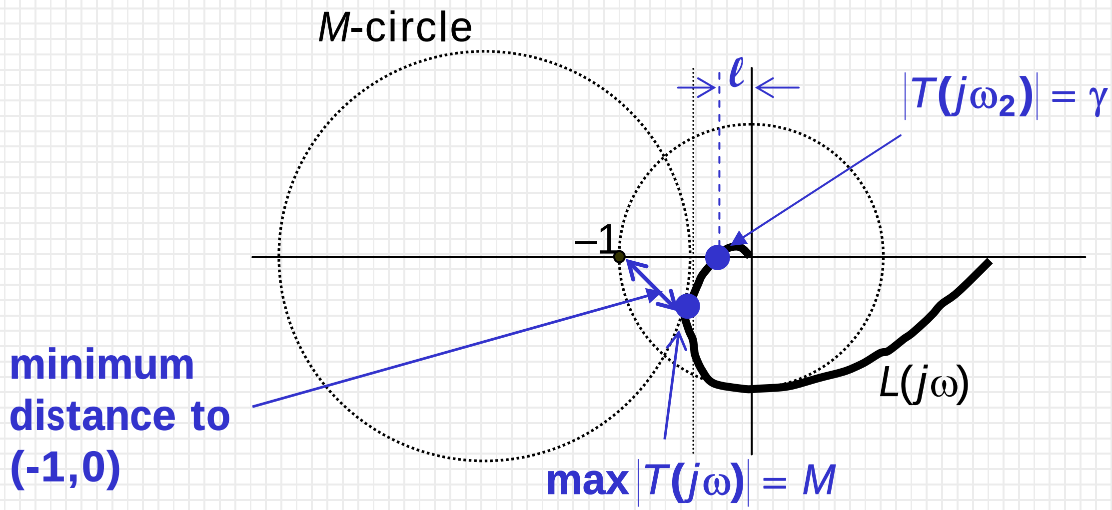
$$\det(I + T\Delta) = \frac{1}{(s+1)^2}\left(s^2 + (2 + \delta_1 + \delta_2)s + 1 + \delta_1 + \delta_2 + \left(1 + a^2\right)\delta_1\delta_2\right).$$

If we take

then the closed-loop system is unstable.

This example should serve as a clear warning that naively using the individual loop margins to judge closed-loop robustness can be misleading. Unfortunately, some have extrapolated this claim to conclude that any technique relying on individual loop closure is therefore flawed.

As it turns out, this example proved a well known fact: gain and phase margins are not a good measure of robustness. We saw that in our study of SISO systems in Chapter 5.

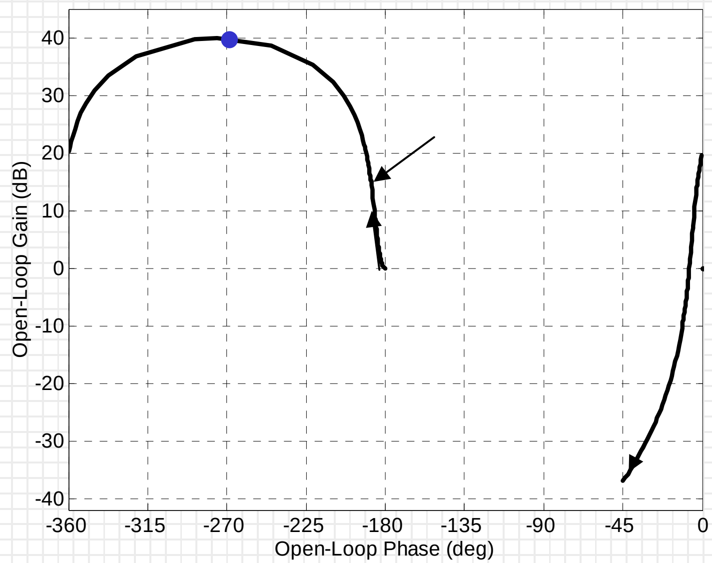


We observed that if the minimal distance to (-1,0) is very small, then very small simultaneous gain and phase variations in $L(j\omega)$ can destabilize the system regardless of its PM and GM.

The same generalizes to MIMO systems. If we close the 2nd loop and do not achieve decent robustness, this weakness will be inherited by the MIMO system.

Let's study this key point in detail. Our sequential closure starts with the 2nd loop so the loop is simply

The loop is shown next.



Note that we have slightly modified the denominator to avoid infinite loop gains. This simplifies our graphically-based analysis.

so

It is obvious that this loop has NO robustness whatsoever in certain directions.

Nevertheless, continuing with our sequential design, the 1st loop becomes

$$p_{11}^{2} \equiv p_{11} - \frac{p_{12}p_{21}c_2}{1+c_2p_{22}} \underset{c_2=1}{=} p_{11} - \frac{p_{12}p_{21}}{1+p_{22}} = \frac{1}{s}.$$

Of course, this loop by itself has a wonderful stability margin property. However, due to MIMO interactions, p_{11} also depends on the p_{ij} and specifically on $(1+p_{22})$. Moreover, the generalized Nyquist stability criterion requires that we deal with

This proves that inferring stability robustness solely based on the plants p_{11}^{2} and p_{22}^{2} is wrong.

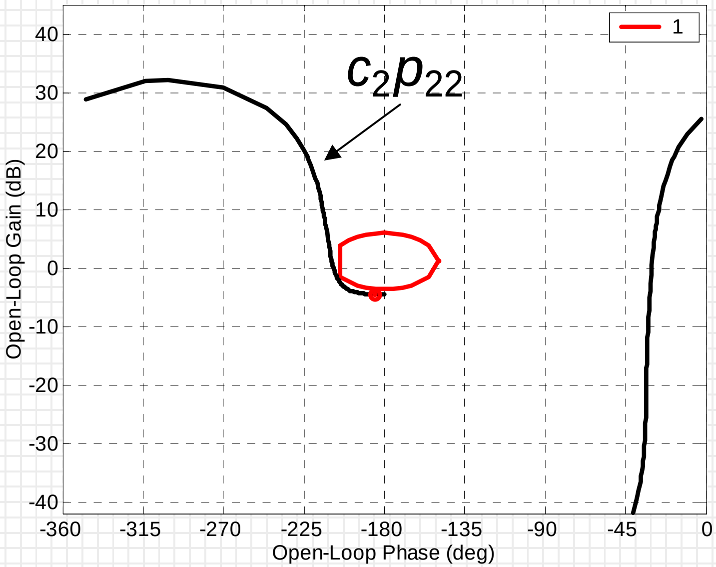
The correct design must include peaking constraints such as, for example (more on this later),

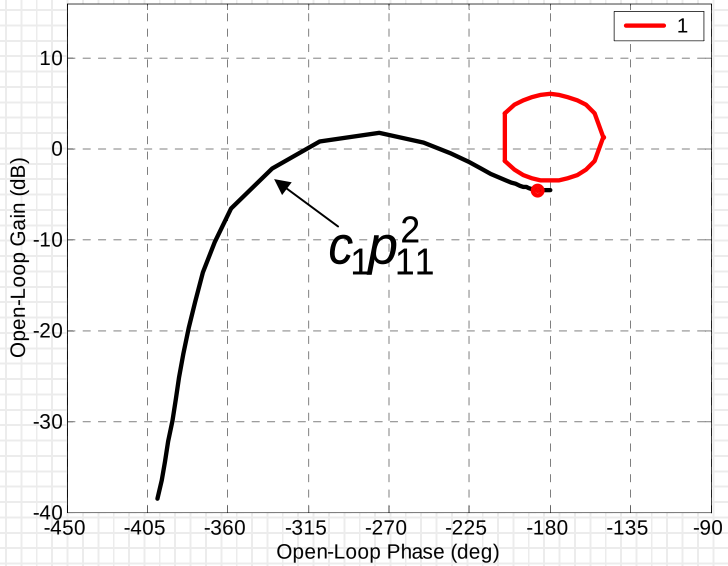
$$\left|I + p_{11}^{2}(j\omega)\right| \geq 0.5, \quad \omega \geq 0$$

$$\left|I + p_{22}(j\omega)\right| \geq 0.5, \quad \omega \geq 0.$$

The loop designs are shown below. Obviously, c_1 and c_2 are no longer unity gains.

$$c_1 = c_2 = \frac{0.2386s^2 + 6.5855s + 40.9931}{s^2 + 16.5720s + 68.6575}$$





$$\min_{\omega} \left| I + PC \right| = \min_{\omega} \left| I + p_{11}^2(j\omega) \right| \left| I + p_{22}(j\omega) \right| \geq 0.5^2.$$

Even though this system was not designed against quantified uncertainty, it offers robustness against certain degree of uncertainty. The uncertainty

$$\delta_1 = 0.9, \quad \delta_2 = 1.1$$

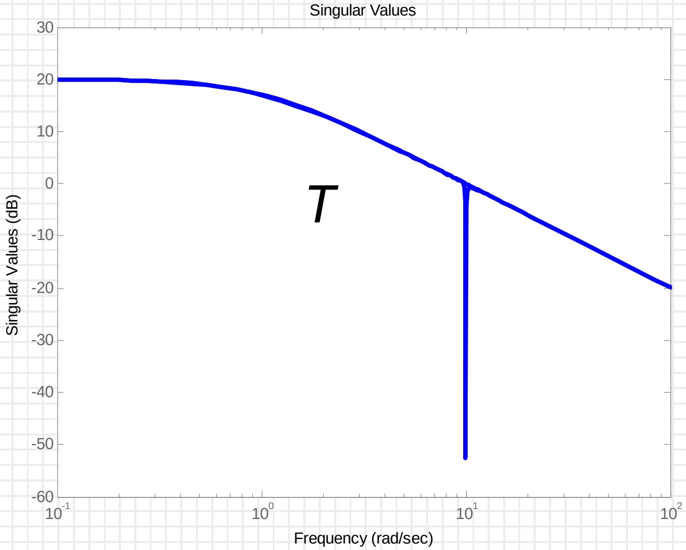
that destabilized the plant compensated with unity gain controller, does not destabilize the present design.

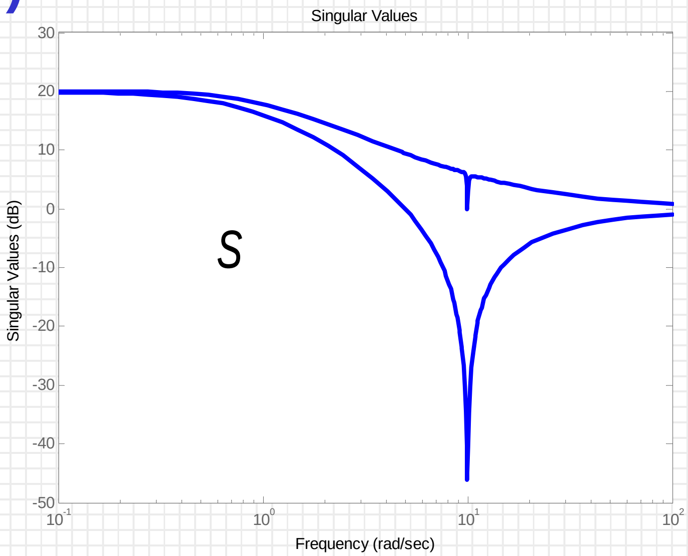
Of course, we can construct new uncertainty that will destabilize even this design. Or any other design for that matter (irrelevant of the control design technique). Robustness is achieved with respect to quantified uncertainty.

A good measure of robustness (not always) are the singular values of closed-loop MTFs. Maximal singular values below 1 generally indicate good robustness whereas peaking above 1 generally indicates robustness issues.

The singular values for *S* and *T* in our example (*C* = *I*) exhibit peaking of 10 at very low frequencies.

sigma()

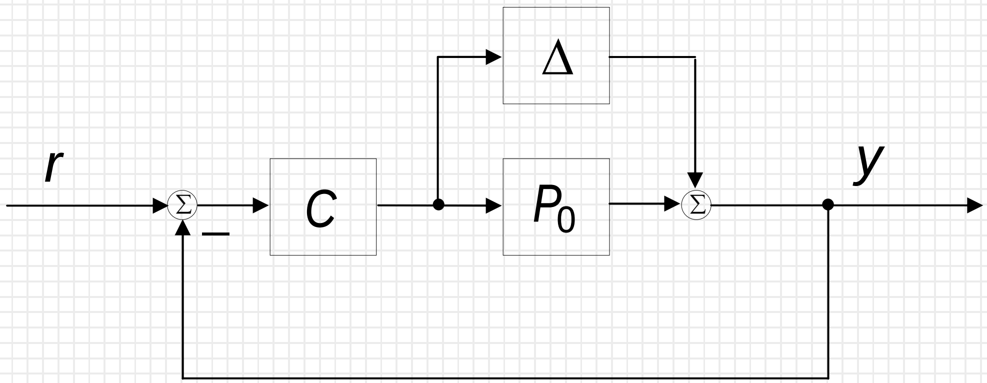




16.2. Design For Stability Margins

The robustness problem discussed in the previous section has a more formal treatment. For example, suppose the plant P has an *additive uncertainty.* Specifically, the plant family is described by an (additive) unstructured uncertainty

$$\mathcal{P} = \left\{ P(s) = P_0(s) + \Delta(s) : \Delta(s) \text{ stable}, \ \|\Delta\|_\infty < \gamma \right\}$$

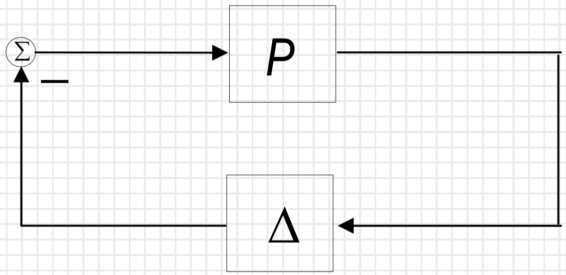


It can be shown that if the nominal closed-loop system is stable, then we have robust stability iff

Note that uncertainty Δ MTF can be real or complex, diagonal or fully populated matrix.

This robust stability result is related to the small gain theorem.

Theorem. Consider the system shown below where *P* and Δ are rational, stable MTFs.



The closed-loop system is internally stable iff

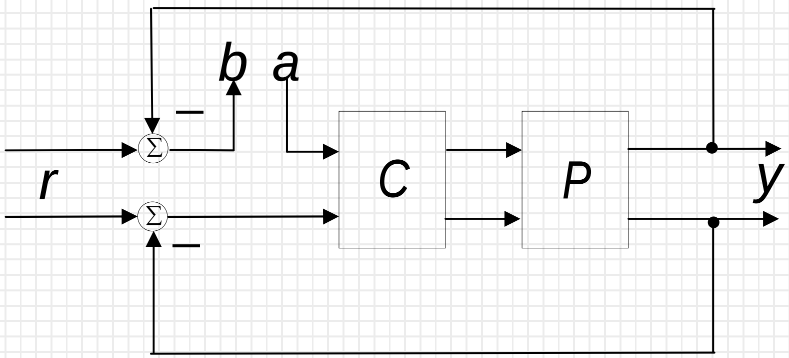
This a strong result which does depend on the sign of feedback as long as the "gain" of the uncertainty is bounded relative to the plant "gain".

The H_∞ technique focuses on design of controllers for the class of uncertainty models as in the previous page to achieve robust stability and robust performance.

QFT aims at control design for quantified, structured uncertainty with additional robustness measures as discussed next.

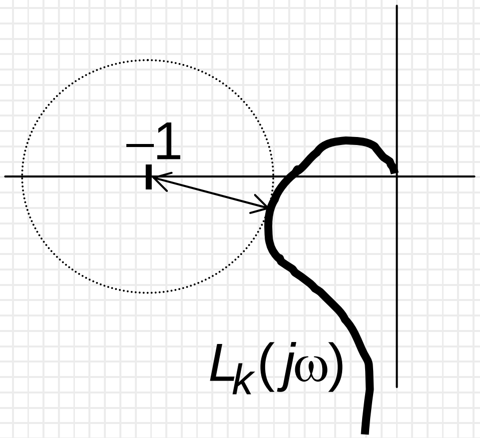
In addition to requiring that the plot of det($I+PC$) has the correct number of crossings necessary for stability, and that it stays away from a disk about the origin, it is suggested that MIMO loops also posses good margins in the following sense.

Consider a MIMO feedback system with all loop closed except for the k'th loop.



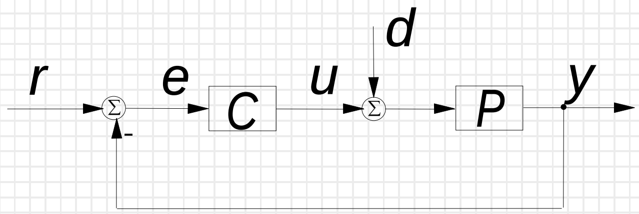
When this loop is opened between a-b, the (SISO) loop transmission L_k is measured from input a to output b. The MIMO margin of this loop is defined as

The interpretation of this margin is the same as in the SISO case: the minimal distance between $L(j\omega)$ and (-1,0).



The difference from SISO margin is that L_k depends not only on p_{kk}, but on the multivariable plant and remaining c_i controllers. Hence, it is essential to achieve decent margins at all loops. Moreover, depending on the problem at hand, additional margins at other loop breakage locations may have to be enforced.

It can be shown that the same L_k appears in SISO-like closed-loop relations such as



In a 2x2 feedback system, L_1 can be computed by closing the 2nd loop then deriving the equivalent 1st loop plant. This can be done using the inverse-based or direct design procedures. This is discussed next.

Using a direct procedure (assuming a 2x2 system and that we close sequentially in order), and applying Gauss elimination

$$\det(I+PC) = (1+c_1 p_{11})(1+c_2 p_{22}^2)$$

where

$$p_{22}^2 = p_{22} - \frac{p_{12}p_{21}c_1}{1+c_1 p_{11}} = \frac{p_{22}+|P|c_1}{1+c_1 p_{11}}$$

We observe that we achieve stability iff we stabilize $c_2 p^2_{22}$ regardless if $c_1 p_{11}$ has been stabilized (why?).

Also note that the plant in the inverse-based procedure at the 2nd step (see CH 15) is

$$\pi^2_{22} = \pi_{22} - \frac{\pi_{12}\pi_{21}}{c_1 + \pi_{11}} = \frac{c_1 \pi_{22} + |\pi|}{c_1 + \pi_{11}} = \frac{c_1 \frac{p_{11}}{|P|} + \frac{1}{|P|}}{c_1 + \frac{p_{22}}{|P|}} = \frac{1 + c_1 p_{11}}{p_{22} + |P| c_1}$$

Hence, stabilizing the 2nd loop in the direct design is equivalent to stabilizing the 2nd loop in the inverse- based design!

Moreover, the actual plant in the 1st loop is

$$p^2_{11} = p_{11} - \frac{p_{12} p_{21} c_2}{1 + c_2 p_{22}} = \frac{p_{11} + |P| c_2}{1 + c_1 p_{22}} =$$

That is, when designing via inverse-based procedure, we are assuming infinite gain in the 2nd loop.

Back to the margin problem. We have seen that the equivalent plant for the k'th loop with all other loops closed is the same in both inverse-based and direct procedures.

The first loop L_1 is given by

$$L_1 = c_1 p_{11}^2 = \frac{c_1}{\pi_{11}^2}.$$

Similarly, L_2 is given by

$$L_2 = c_2 p_{22}^2 = \frac{c_2}{\pi_{22}^2}.$$

But what about designing for margins at the 1st step where c_2 is unknown? Specifically,

$$\left|1 + c_1 p_{11}^2\right|^{-1} = \left|1 + \frac{1 + c_1(p_{11} + c_2|P|)}{1 + c_2 p_{22}}\right|^{-1}$$

$$= \left|\frac{1 + c_1 p_{11} + c_2(p_{22} + c_1|P|)}{1 + c_2 p_{22}}\right|^{-1}$$

$$= \left|\frac{A + Bc_2}{C + Dc_2}\right| \le m_1.$$

We must watch for designing c_1 in a manner that would not require c_2 to have infinite bandwidth at high frequencies. Namely, the margin spec has the form

$$\left|1+c_1 p_{11}^2\right|^{-1} \le m_1.$$

We implement the above by insuring that c_2 is not required to have high gain beyond the useful bandwidth.

In terms of bounds, the above can be written as

$$w = \frac{A+Bc_2}{C+Dc_2} \in \text{ disk of radius } m_i \text{ about the origin}$$

The inverse map is given by

Strictly properness of c_2 implies that at high frequencies the above map should contain the origin,

$$0 \in B - Dw.$$

For the margin constraint to be consistent with c_2 not required to have high gain beyond the useful bandwidth, we obtain a necessary condition

which, in term of the actual variables, implies

$$\left|\frac{B}{D}\right| = \left|\frac{p_{22}}{p_{22} + c_1|P|}\right| = \left|\frac{1}{1 + c_1 \frac{|P|}{p_{22}}}\right| = \left|\frac{1}{1 + c_1 \frac{1}{\pi_{11}}}\right| \le m_1 \quad \omega \ge \omega_{bw}.$$

On the other hand, we should not constrain the bandwidth of c_2. Hence,

In term of the actual variables we have another condition

$$\left|\frac{A}{C}\right| = \left|\frac{1}{1 + c_1 p_{11}}\right| \le m_1 \quad \omega \ge \omega_{bw}.$$

In is interesting to note that the same two constraints can be arrived at by

$$\left|1+p_{11}^{2}c_1\right| = \left|\frac{1+c_2p_{22}}{1+c_1p_{11}+c_2(p_{22}+c_1|P|)}\right| \rightarrow$$

In the general *nxn* case, as shown earlier, at the *k*'th loop closure step our problem becomes

$$(I+PC)^{-1} = (I+P(C-C_k^x)+PC_k^x)^{-1} = \frac{A+c_kB}{1+c_k\delta}$$

where

(for a total number of combinations of 2^{n-k}).

Back to the 2x2 case. At the final step, c_2 must be designed to satisfy the margin weight at the 2nd loop, as well as to not “mess up” the 1st loop. This amount to two inequalities for margin specs

$$\left|\frac{1}{1+L_1(c_2)}\right| \leq m_1$$

$$\left|\frac{1}{1+L_2(c_2)}\right| \leq m_2$$

which are standard bilinear maps that can be written in terms of the plant or its inverse

Recall the example from section 12.2. The lesson learned was that it is not sufficient to check loop margins alone. We must also work with $\det(I+PC)$, so a 3rd constraint is

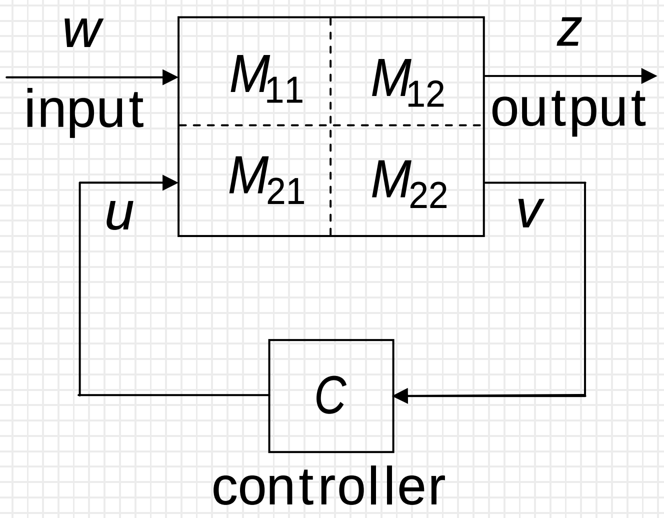
In practice,

so this constraint is rarely verified.

16.3. LFTs

Operator-valued linear fractional transformations (LFT) are used in modern control (e.g., H_∞). This power of this formulation can also be exploited in MIMO QFT as follows (*Yaniv*, 1999).

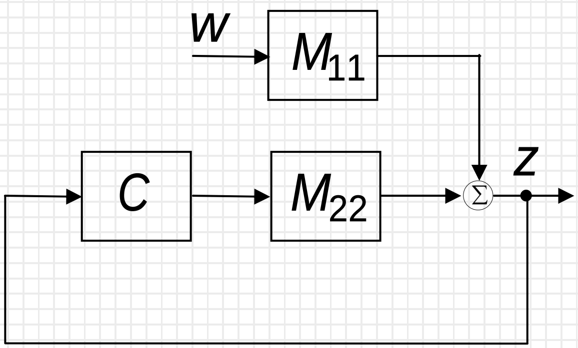
A general MIMO feedback in an LFT form is shown below.



$$
\begin{aligned}
z &= M_{11}w + M_{12}u \\
v &= M_{21}w + M_{22}u \\
u &= Cv
\end{aligned}
$$

Closing loops and evaluating input-output relations gives

For example, a weighted plant output disturbance rejection is modeled by



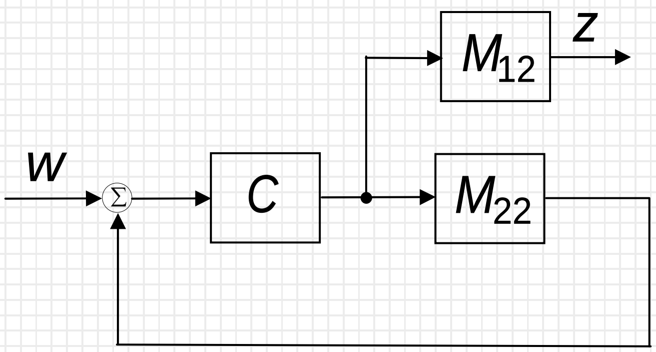
$$M_{11} = M_{21}$$

$$M_{12} = M_{22}$$

$$z = M_{11}w + M_{22}C(I - M_{22}C)^{-1}M_{11}w$$

For $M_{11} = M_{22}$, we have a weighted plant output disturbance problem.

The control effort in a tracking problem is modeled by



$$M_{11} = 0$$

To derive bi-linear maps for tuning purposes (Section 15.1), recall the notation

$$\begin{aligned} C_k &= \mathrm{diag}[0,\ldots,c_k,0,\ldots,0] \\ F_k &= I + P(C - C_k) \\ u_k &= [p_{1k},\ldots,p_{mk}]^T \\ v_k &= [0,\ldots,1,0,\ldots,0]^T,\ 1 \text{ in position } k. \end{aligned}$$

To tune c_k, first split the MTF

$$\begin{aligned} T_{zw} &= M_{11} + M_{12}C(I - M_{22}C)^{-1}M_{21} \\ &= M_{11} + M_{12}C_k(I - M_{22}C)^{-1}M_{21} + M_{12}(C - C_k)(I - M_{22}C)^{-1}M_{21} \end{aligned}$$

then

$$\begin{aligned} (I - M_{22}C)^{-1} &= (I - M_{22}(C - C_k)\text{-}M_{22}C_k)^{-1} \\ &= (F_k - c_k u_k v_k^{\mathrm{T}})^{-1} \\ &= \frac{F_k^{-1} + c_k F_k^{-1}(u_k v_k^{\mathrm{T}} F_k^{-1}\text{-}v_k^{\mathrm{T}} F_k^{-1} u_k I)}{1\text{-}c_k v_k^{\mathrm{T}} F_k^{-1} u_k} \\ &\equiv \frac{A_k + c_k B_k}{1 + c_k \delta} \end{aligned}$$

$$\begin{aligned} C_k &= \mathrm{diag}[0,\ldots,c_k,0,\ldots,0] \\ F_k &= I + P(C - C_k) \\ u_k &= [p_{1k},\ldots,p_{mk}]^T \\ v_k &= [0,\ldots,1,0,\ldots,0]^T, \ 1 \text{ in position } k. \end{aligned}$$

where

$$F_k = I - M_{22}(C - C_k).$$

Nothing that

$$C_k(I-M_{22}C)^{-1}=\begin{cases}0 & \text{any row } \neq k\\ \dfrac{c_kA_k(k,:)}{1+c_kA_k(k,:)} & \text{otherwise}\end{cases}$$

and

$$C_kB_k=0$$

(k'th row of B_k is zero), so

$$M_{12}(C-C_k)(I-M_{22}C)^{-1}M_{21}=\frac{c_kM_{12}(:,k)A_k(k,:)}{1+c_k\delta}M_{21}$$

and finally

$$\begin{aligned}T_{zw}&=M_{11}+M_{12}C(I-M_{22}C)^{-1}M_{21}\\&=M_{11}+M_{12}(C-C_k)\frac{A_k+c_kB_k}{1+c_k\delta}M_{21}+\frac{c_kM_{12}(:,k)A_k(k,:)}{1+c_k\delta}M_{21}.\end{aligned}$$

A special case occurs when some of the loops are closed with infinite bandwidth. In this case we factor the controller using

$$C = UV$$

$$U = \text{diag}(u_i), \quad u_i = \begin{cases} c_i & c_i \neq \infty \\ 1 & c_i = \infty \end{cases}$$

$$V = \text{diag}(v_i), \quad v_i = \begin{cases} 1 & c_i \neq \infty \\ \infty & c_i = \infty \end{cases}.$$

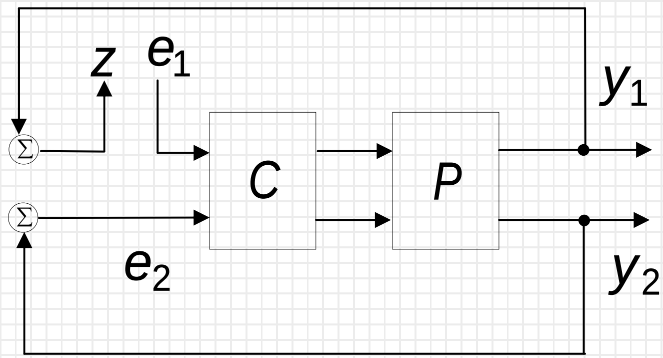
The resulting LFT becomes

$$T_{zw} = M_{11} + M_{12}C(I - M_{22}C)^{-1}M_{21} = M_{11} + M_{12}U(\text{V}^{-1} - M_{22}U)^{-1}M_{21}.$$

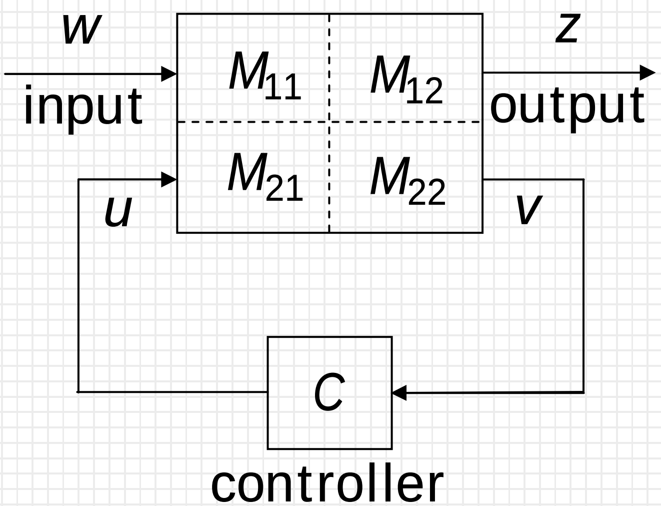
Note that

$$V^{-1} = \text{diag}(v_i^{-1}), \quad v_i^{-1} = \begin{cases} 1 & c_i \neq \infty \\ 0 & c_i = \infty \end{cases}.$$

LFTs are useful for computing equivalent open-loop functions when the loop *k* is broken somewhere with all other loops closed. For example, in a 2x2 system, breaking the loop as shown below.



The standard generalized plant structure is **(note the positive feedback)**



where

$$w = e_1$$

$$v = y_2$$

$$u = e_2$$

and $y = PCe$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} p_{11}c_1 & p_{12}c_2 \\ p_{21}c_1 & p_{22}c_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

The governing generalized plant equations are

$$\begin{aligned} z &= M_{11}w + M_{12}u \\ v &= M_{21}w + M_{22}u \\ u &= Gv \end{aligned}$$

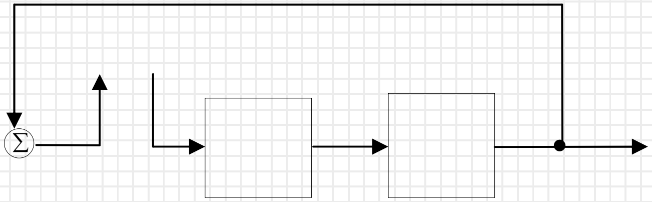
The I/O LFT is

$$z = M_{11}w + M_{12}G(I - M_{22}G)^{-1}M_{21}w \equiv T_{zw}w$$

so

$$\begin{aligned}
T_{zw} &= p_{11}c_1 + (p_{12}c_2)(1)(1\text{-}p_{22}c_2(1))^{-1}(p_{21}c_1) \\
&= \left(p_{11} + \frac{p_{12}c_2p_{21}}{1 - p_{22}c_2}\right)c_1 \\
&= \frac{p_{11} - |P|c_2}{1 - p_{22}c_2}c_1 \quad \left(\text{negative feedback} \Rightarrow \frac{p_{11} + |P|c_2}{1 + p_{22}c_2}c_1\right) \\
&= p_{11}^2 c_1 \quad (L_1 \text{ in Yaniv's book}).
\end{aligned}$$

We now have



Note that due to diagonality of C we can pull out the loop's siso controller as follows

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12}c_2 \\ p_{21} & p_{22}c_2 \end{bmatrix} \begin{bmatrix} c_1 e_1 \\ e_2 \end{bmatrix}$$

so

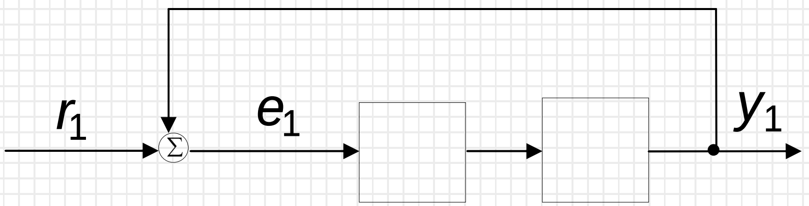
$$M = \left[\begin{array}{c|c} p_{11} & p_{12}c_2 \\ \hline p_{21} & p_{22}c_2 \end{array}\right], \quad G = 1$$

and the open-loop has the same relation

$$\frac{z}{c_1 e_1} = T_{zw} = p_{11} + (p_{12}c_2)(1)(1\text{-}p_{22}c_2(1))^{-1}(p_{21})$$

$$= p_{11} + \frac{p_{12}c_2 p_{21}}{1 - p_{22}c_2} \equiv p_{11}^2.$$

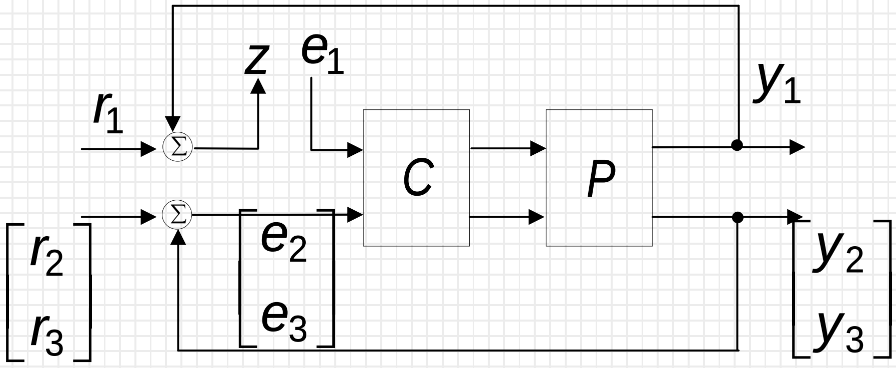
The relation between r_1 and y_1 is therefore



and we have shown that

$$\frac{y_1}{r_1} = \frac{c_1 p_{11}^2}{1 + c_1 p_{11}^2} \triangleq \frac{L_1}{1 + L_1}.$$

In a 3x3 plant



$$M = \left[\begin{array}{c|cc} p_{11} & p_{12}c_2 & p_{13}c_3 \\ \hline p_{21} & p_{22}c_2 & p_{23}c_3 \\ p_{31} & p_{32}c_2 & p_{33}c_3 \end{array}\right]$$

$$G = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The I/O LFT is

$$\frac{x}{c_1e_1} = T_{zw} = p_{11} + \begin{bmatrix} p_{12}c_2 & p_{13}c_3 \end{bmatrix}[I]\left[I - \begin{bmatrix} p_{22}c_2 & p_{13}c_3 \\ p_{32}c_2 & p_{33}c_3 \end{bmatrix}[I]\right]^{-1}\begin{bmatrix} p_{21} \\ p_{31} \end{bmatrix}$$

which is easily computed using LTI objects (with positive feedback)

```
C(1,1) = tf(1,1)% note output is c1e1
L = P*C;
M11 = L(1,1);
M12 = L(1,2:3);
M21 = L(2:3,1);
M22 = L(2:3,2:3);
p11e = M11+M12*(eye(2,2))*inv(eye(2,2)-M22))*M21;
```

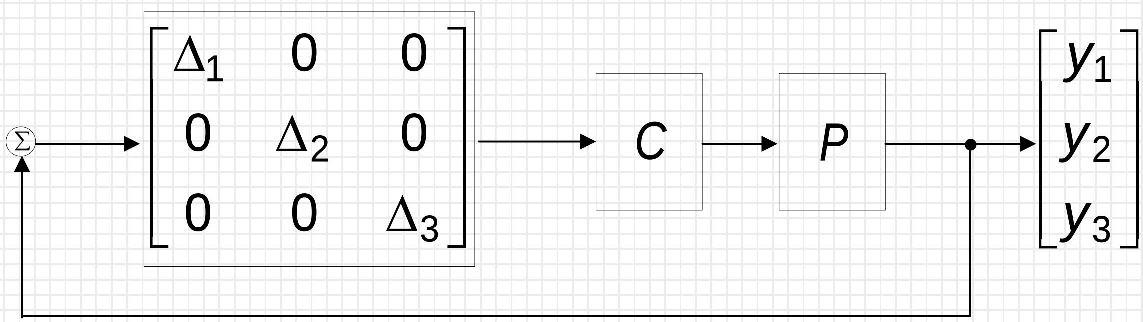

This siso transfer function is the (1,1) element in the MTF T

$$Y = T_I r = (I + PC)^{-1} PCr.$$

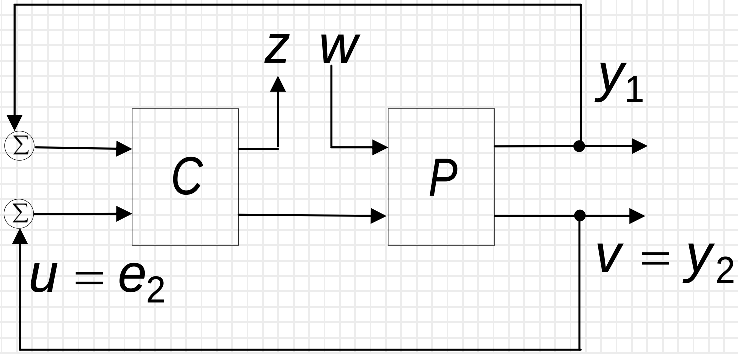
Invoking peaking constraints on the diagonal t_{11} amounts to inserting uncertainty (disk type) between x and e_1 with the constraint

$$\left| \frac{L_1}{1 - L_1} \right| \le m_1$$

It is always recommended to require robustness against diagonal unstructured uncertainty of the above form. It should be inserted in loop locations that correlate to practical reasons. In the above example



Let us repeat the same exercise with the loop broken at a different location as shown below.



$$
\begin{aligned}
y_1 &= p_{11}w + p_{12}c_2u \\
v &= p_{21}w + p_{22}c_2u \\
u &= v
\end{aligned}
$$

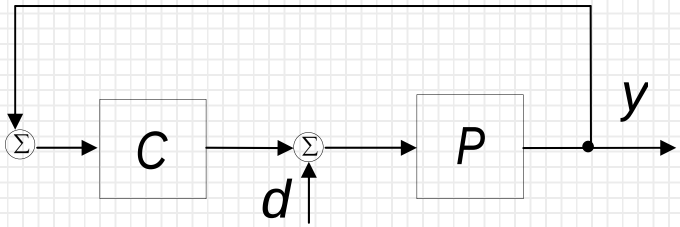
and since $z = c_1 y_1$

$$
\begin{aligned}
z &= p_{11}c_1w + p_{12}c_1c_2u \\
v &= p_{21}w + p_{22}c_2u \\
u &= v
\end{aligned}
$$

so

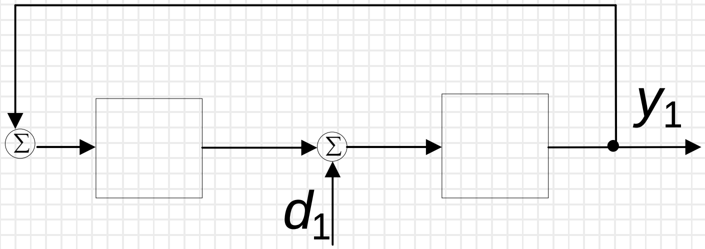
$$T_{zw} = p_{11}c_1 + (p_{12}c_2c_1)(1\text{-}p_{22}c_2)^{-1}(p_{21}) = \frac{p_{11} - |P|c_2}{1 - p_{22}c_2}c_1.$$

A plant input disturbance problem has the form



$$y = (I - PC)^{-1} Pd$$

The scalar function from d_1 to y_1 has the block diagram shown below



$$p_{11}^2 \equiv T_{zw}$$

with

$$\frac{y_1}{d_1} = \frac{p_{11}^2}{1 - c_1 p_{11}^2} = \frac{p_{11} - |P| c_2}{1 - p_{22} c_2 - p_{11} c_1 + |P| c_1 c_2}.$$

Whereas computing the relation directly from the MTF gives (using negative feedback)

$$\begin{bmatrix} \pi_{11}^2 + c_1 & 0 \\ \pi_{12} & \pi_{22} + c_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-\pi_{12}}{\pi_{11}+c_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$\frac{y_1}{d_1} = \frac{1}{\pi_{11}^2 + c_1} = \frac{1}{\left(\pi_{11} - \frac{\pi_{12}\pi_{21}}{c_2+\pi_{22}}\right) + c_1}$$

$$= \frac{1}{\frac{\pi_{11}c_2 + \pi_{11}\pi_{22} - \pi_{12}\pi_{21}}{c_2+\pi_{22}} + c_1} = \frac{c_2 + \frac{p_{11}}{|P|}}{\frac{p_{22}}{|P|}c_2 + \frac{1}{|P|} + \left(c_2 + \frac{p_{11}}{|P|}\right)c_1}$$

$$= \frac{p_{11} + |P|c_2}{1 + p_{22}c_2 + p_{11}c_1 + |P|c_1c_2}.$$

Which is the same as the relation obtained earlier with the exception for minus signs for positive feedback.