15. Multivariable QFT Design (Part 2)

15.1. Reducing Conservatism. Recall that in the input disturbance rejection problem of Ch. 12, with the exception of the last step, the design algorithms leads to conservatism. This was a result of over-bounding closed-loop interactions from the loops yet to be designed. We now consider two ways to minimize this over-design.

Low Frequency Performance Bounds. Assuming "large" control gains at low frequencies (below crossover), we approximate the input-output relations by

which leads to simple performance inequalities of the form

$$|y_i(j\omega)| \le \alpha_i(\omega) \quad \forall P \in \mathcal{P} \text{ and } d \in \mathcal{d}$$

The above direct form is quite different from the performance inequality developed in Ch. 14 for the first loop

$$\left|y_1(j\omega)\right| \leq \frac{|d_1|+|\pi_{12}y_2|}{|\pi_{11}+c_1|} \leq \frac{|d_1|+\alpha_2|\pi_{12}|}{|\pi_{11}+c_1|} \leq \alpha_1(\omega),\ \forall P \in \mathcal{P} \text{ and } d \in \mathcal{d}.$$

The exact amount of achieved over-design reduction varies from one problem to another. However, the computational complexity of the associated QFT bound is reduced. Also, the above approximation is effective only at the low frequency range where we have large loop gains. At the mid frequency range, margin bounds should dominate loop shaping constraints.

Tuning. Recall the design in the Chap. 14. Closed-loop performance of the 1st loop (i.e., $|y_1|$) exhibits the expected over-design. When C is completely known, is it possible to tune c_1 such that over-design is reduced without adversely affecting margins and performance of the MIMO system? It turns out that by exploiting multivariable directionality, this is often feasible. In fact, we now show that each relation from the j'th input d_j to the i'th output y_i in terms of c_k has a bi-linear form.

Before we proceed, we need the following relation (see *Yaniv*, 1999):

$$(F + uv^T)^{-1} = F^{-1} - \frac{F^{-1}uv^TF^{-1}}{1 + v^TFu}.$$

Let

$$\begin{aligned} C_k &= \mathrm{diag}[0,\ldots,c_k,0,\ldots,0] \\ F_k &= I + P(C - C_k) \\ u_k &= [p_{1k},\ldots,p_{mk}]^T \\ v_k &= [0,\ldots,1,0,\ldots,0]^T,\ 1 \text{ in position } k. \end{aligned}$$

Using this relation

$$S_o = (I+PC)^{-1} = (I + P(C - C_k) + PC_k)^{-1} = (F_k + c_k u_k v_k^T)^{-1}$$

$$= F_k^{-1} - \frac{c_k F_k^{-1} u_k v_k^T F_k^{-1}}{1 + c_k v_k^T F_k^{-1} u_k}$$

$$= \frac{F_k^{-1} + c_k F_k^{-1}(v_k^T F_k^{-1} u_k I - u_k v_k^T F_k^{-1})}{1 + c_k v_k^T F_k^{-1} u_k}$$

That is, using $A_k = [a_{ij}]$ and $B_k = [b_{ij}]$, we have

$$\frac{A_k + c_k B_k}{1 + c_k \delta} = \begin{bmatrix} \frac{a_{11} + c_k b_{11}}{1 + c_k \delta} & \cdots & \frac{a_{1m} + c_k b_{1m}}{1 + c_k \delta} \\ \vdots & \cdots & \vdots \\ \frac{a_{m1} + c_k b_{m1}}{1 + c_k \delta} & \cdots & \frac{a_{mm} + c_k b_{mm}}{1 + c_k \delta} \end{bmatrix}.$$

Our plant input disturbance rejection problem has this form

$$y = (I + PC)^{-1} P d$$

For example, in our 2x2 example, to tune c_1, we need two performance inequalities since it affects the performance of both y_1 and y_2. Specifically,

$$d = \begin{bmatrix} d_1 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ d_2 \end{bmatrix}$$

we end up with a set of 4 robust performance inequalities, 2 for each output:

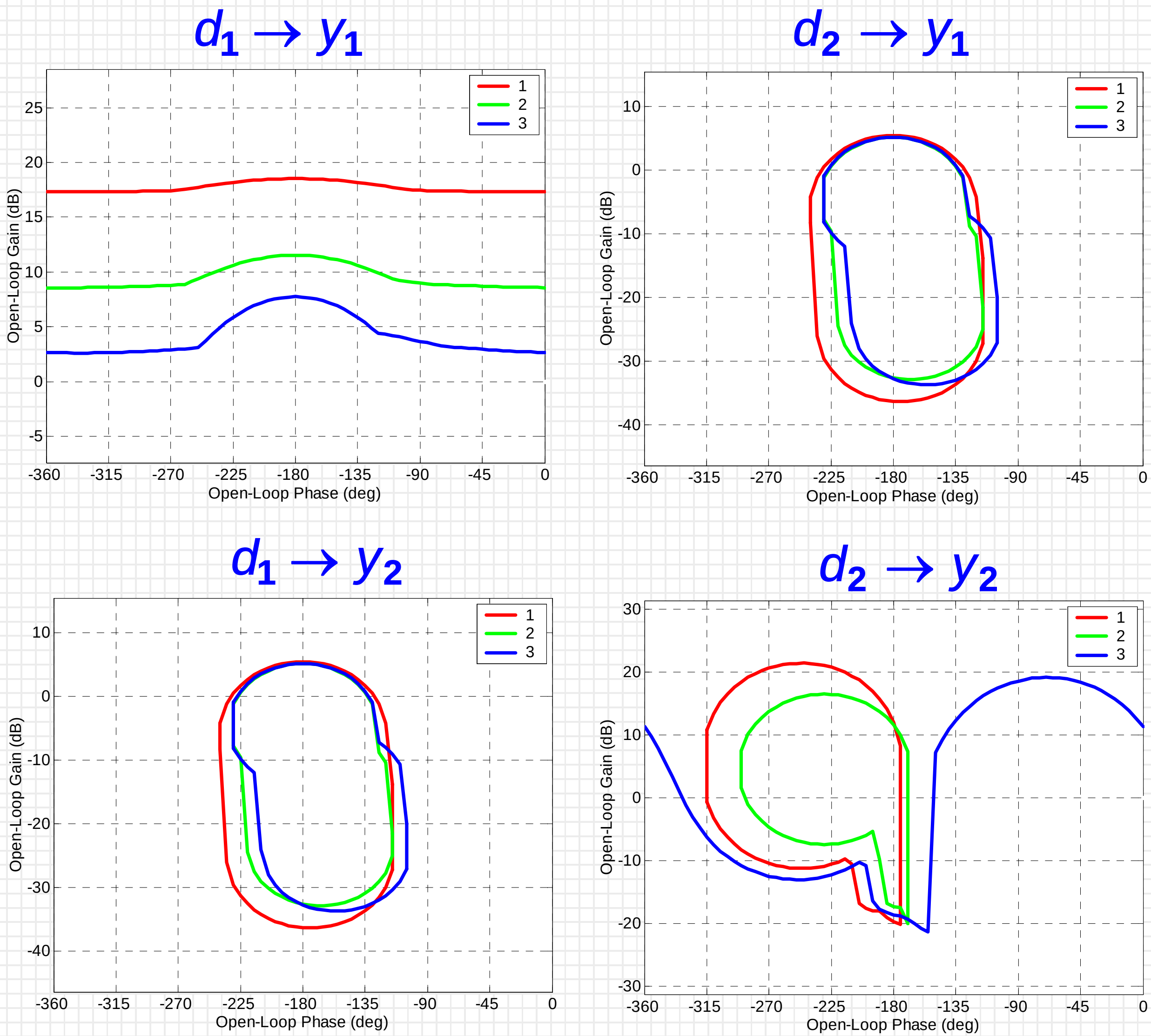
$$|y_1(j\omega)| \le \alpha_1(\omega), \quad \forall P \in \mathcal{P} \text{ and } d \in \mathcal{d}$$

$$|y_2(j\omega)| \le \alpha_2(\omega), \quad \forall P \in \mathcal{P} \text{ and } d \in \mathcal{d}.$$

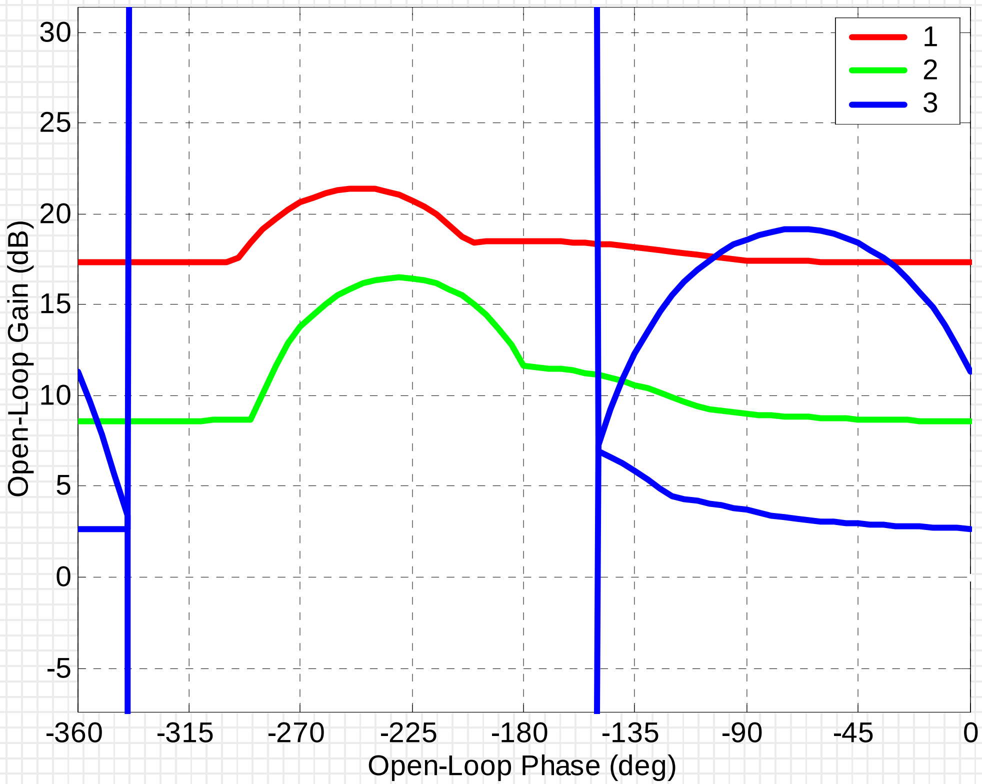
To compute bounds, we need to work with the nominal plant in the 1st loop. Since the 2nd loop is closed, we have

Per our stability criterion, robust stability of the MIMO system is achieved iff c_1 robustly stabilizes $1/\pi^2_{11}$.

The four sets of disturbance rejection bounds are shown next (`ch15_ex1_tune.m`).


$d_1 \to y_1$
$d_2 \to y_1$
$d_1 \to y_2$
$d_2 \to y_2$
1
2
3
Open-Loop Gain (dB)
Open-Loop Phase (deg)
-360
-315
-270
-225
-180
-135
-90
-45
0
25
20
15
10
5
0
-5
-10
-20
-30
-40
30

The intersection of these bounds is quite interesting.



In addition, we have two margin bounds, one in each loop. Since there's only one unknown, the constraints

$$\left|1+c_1/\pi_{11}^2\right| \geq 0.6, \quad \forall P \in \mathcal{P}, \omega \geq 0$$

$$\left|1+c_2/\pi_{22}^2\right| \geq 0.6, \quad \forall P \in \mathcal{P}, \omega \geq 0$$

are bilinear in terms of c_1. For example

$$1+c_1/\pi_{11}^2 = 1+\frac{c_1}{\pi_{11}-\dfrac{\pi_{12}\pi_{21}}{\pi_{22}+c_2}} = \frac{(\det\pi + c_2\pi_{11}) + c_1(c_2+\pi_{22})}{\det\pi + c_2\pi_{11}}.$$

$$1+c_2/\pi_{22}^2 = 1+\frac{c_2}{\pi_{22}-\dfrac{\pi_{12}\pi_{21}}{\pi_{11}+c_1}} = \frac{(\det\pi + c_2\pi_{11}) + c_1(c_2+\pi_{11})}{\det\pi + c_1\pi_{22}}.$$

However, it turns out that

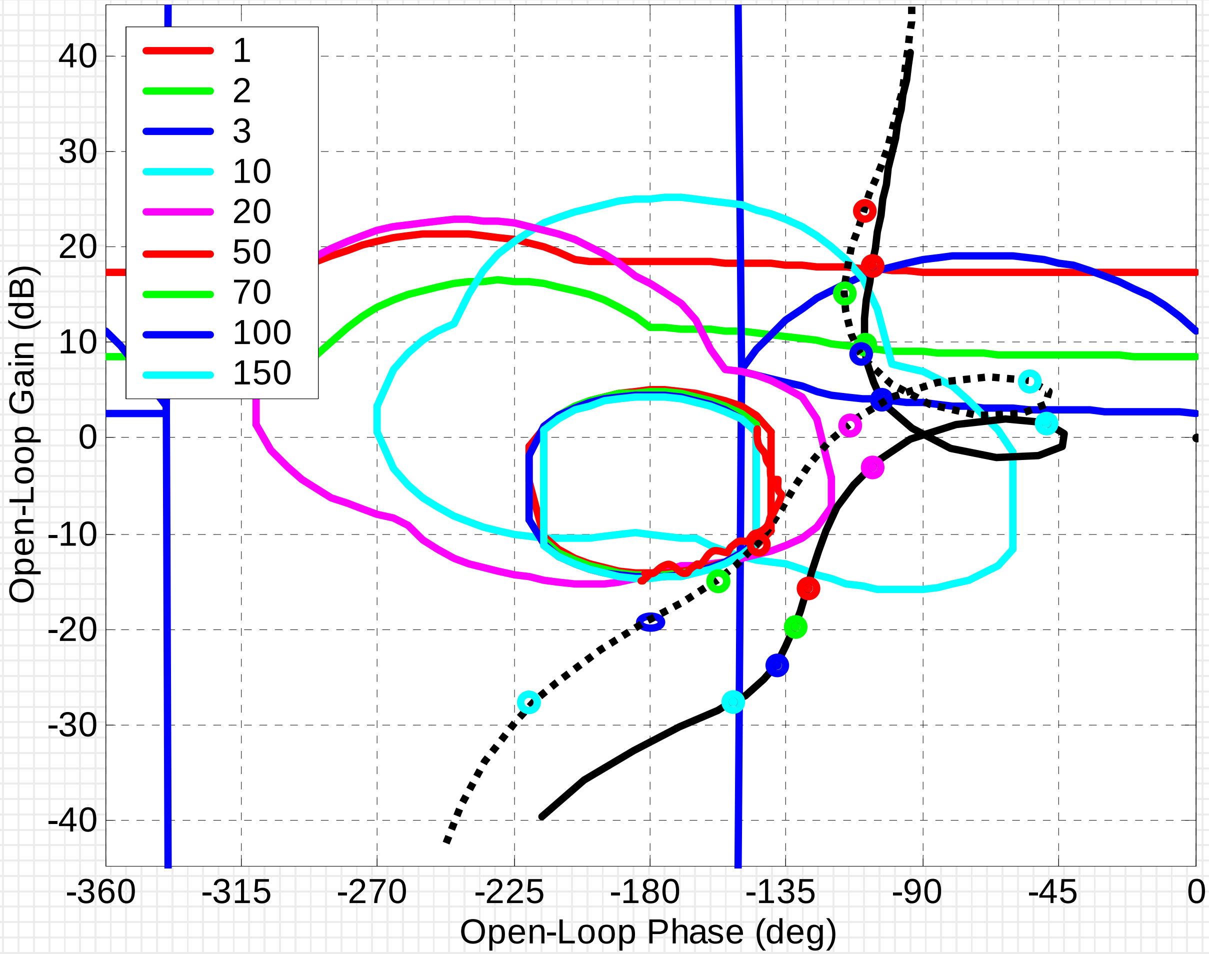
$$S = (I + PC)^{-1} = \begin{bmatrix} \dfrac{1}{1 + c_1 / \pi_{11}^2} & s_{12} \\ s_{21} & \dfrac{1}{1 + c_2 / \pi_{22}^2} \end{bmatrix}$$

And earlier we have shown that

$$(I + PC)^{-1} = \frac{A + c_k B}{1 + c_k \delta_k}$$

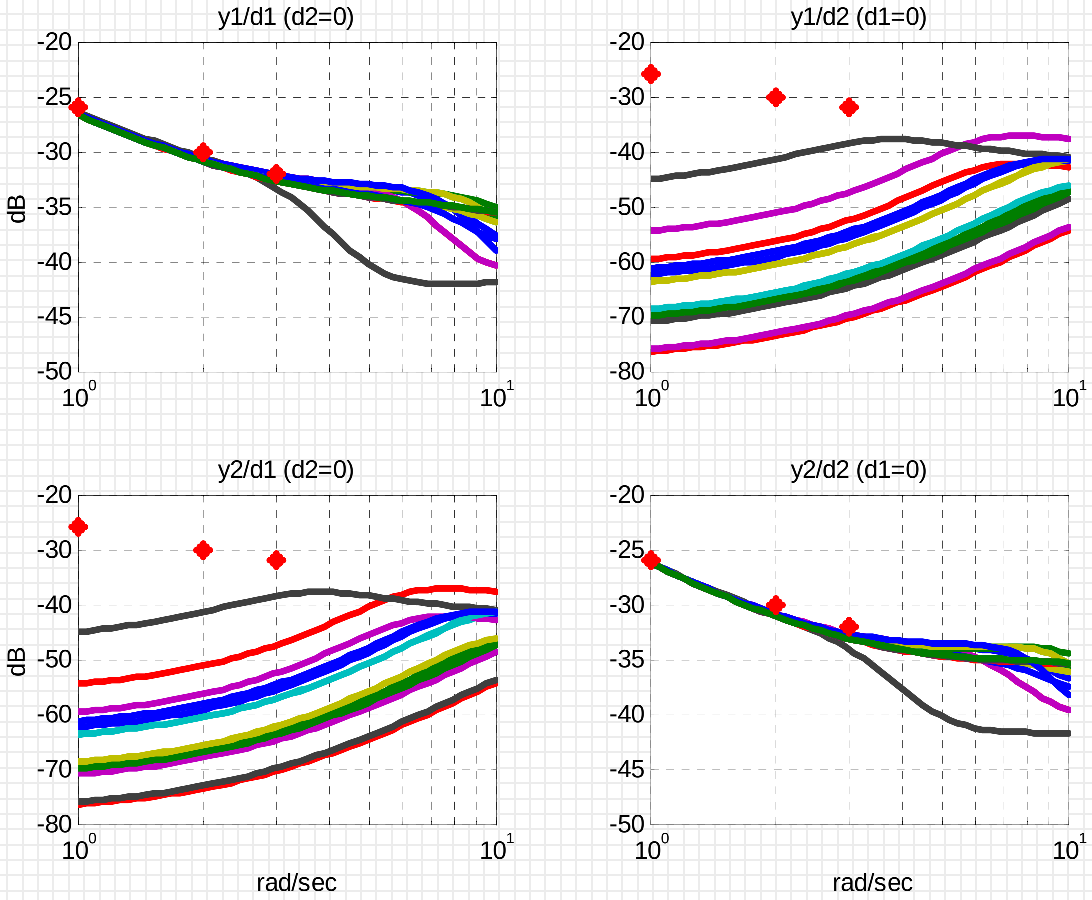
which means that we need not compute the above relations manually prior to computing bounds (see M-file).

Finally, the advantage of tuning is shown below.

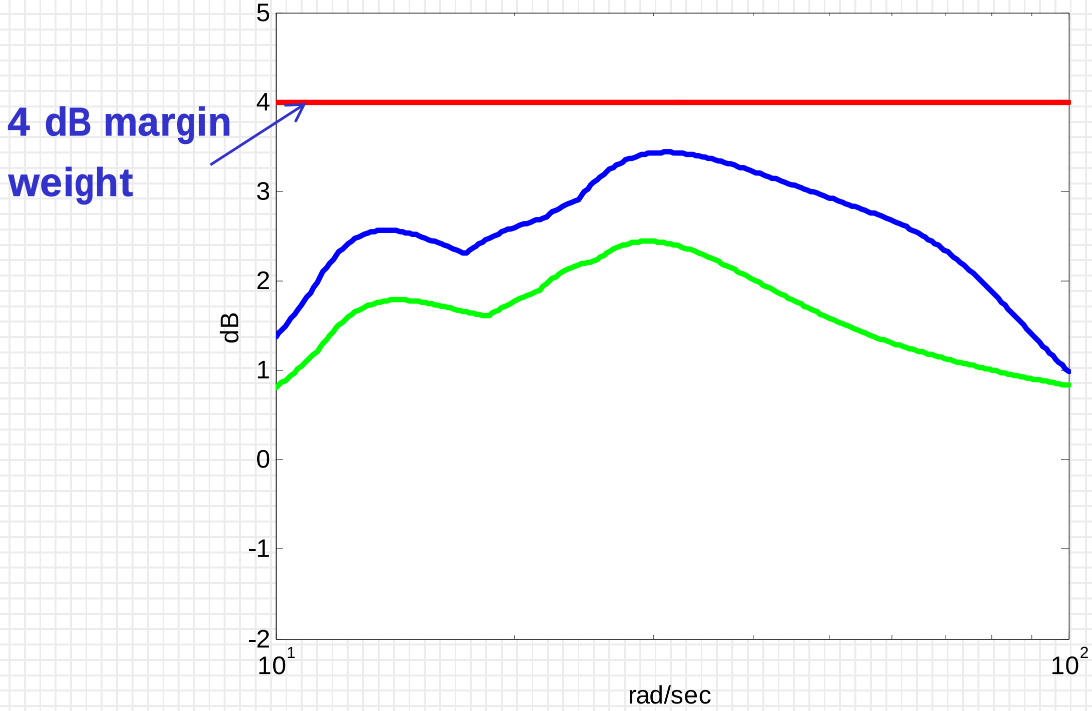


We know c_1 has too large a gain; the trick is to reduce it w/o messing mimo performance!

The over-design is now removed as shown below in the closed-loop simulations.



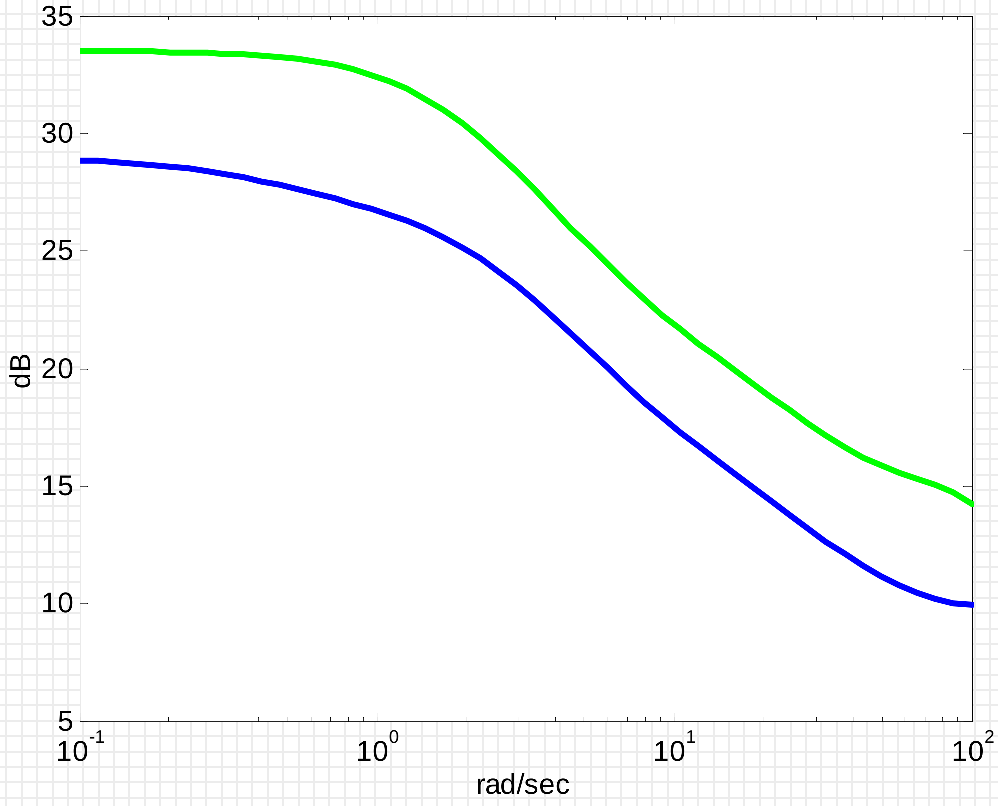
As expected, the margin problem is resolved as well (why?).



$$S = (I + PC)^{-1} = \left[s_{ij} \right]$$

$$s_{ii} = \frac{1}{1 + C_i \frac{1}{\pi_{ii}^2}}$$

Comparison of the siso loop 1 controllers is shown below. Which one is the tuned version?



15.2. Direct and Inverse-Based Design

Stability is established using Nyquist criterion where we count encirclements of det(I+PC). Using inversion

$$\det(I+PC)=\det(P)\det\left(P^{-1}+C\right)$$

and if P is stable, then from Ch. 13

$$P(s)u=\lambda_i(s)u \Rightarrow \text{Nyquist plot of } \det P(s)=\text{Nyquist plot of } \prod_i \lambda_i(s)$$

implying that the Nyquist plot of det(P) does not contribute any encirclements. So let's focus on det(I+PC)

$$\left|P^{-1}+C\right|=\left|\begin{bmatrix}\pi_{11}+c_1 & \pi_{12}\\ \pi_{21} & \pi_{22}+c_2\end{bmatrix}\right|=\left|\begin{bmatrix}1 & 0\\ \frac{-\pi_{21}}{\pi_{11}+c_1} & 1\end{bmatrix}^{-1}\begin{bmatrix}\pi_{11}+c_1 & \pi_{12}\\ 0 & \pi_{22}^2+c_2\end{bmatrix}\right|$$

$$=\left|\begin{bmatrix}\pi_{11}+c_1 & \pi_{12}\\ 0 & \pi_{22}^2+c_2\end{bmatrix}\right|=\left(\pi_{11}+c_1\right)\left(\pi_{22}^2+c_2\right).$$

$$=\left(\pi_{11}+c_1\right)\left(\pi_{22}-\frac{\pi_{12}\pi_{21}}{\pi_{11}+c_1}+c_2\right)$$

$$= (\pi_{11} + c_1)\left(\frac{\pi_{22}\pi_{11} - \pi_{12}\pi_{21} + \pi_{22}c_1}{\pi_{11} + c_1} + c_2\right) = (\pi_{11} + c_1)\left(\frac{\det\pi + \pi_{22}c_1}{\pi_{11} + c_1} + c_2\right).$$

We observe that even if we c_1 does not stabilize p_{11} at the first design step,

and if the term on the right is stable, so is the product. Hence, if c_2 stabilizes the plant at the 2nd step, the MIMO system is stable. Nevertheless, unstable 1st loop adds a burden on c_2.

While not shown here, the same can be said on *nxn* MIMO systems. If c_2 stabilizes the plant at the *n*th (last) step, the MIMO closed-loop system is stable even if earlier loops are not stable. Implicit here are the assumptions of no unstable pole-zero cancellations and no unstable decentralized hidden modes.

The direct scheme (DS) was developed[1] to avoid the need for plant inversion. It is shown there that a similar sequential design procedure is feasible. In certain situation where the plant is unstable with nmp zeros (or delay), the inversion-based scheme (IS) is not desired.

Design for performance in either scheme is similar and not studied here. However, there are interesting connections between the two in terms of stability. This is studied next.

Again, closed-loop stability is established from the Nyquist plot of det(I+PC). Using the sequential procedure we have

$$|I+PC| = \left| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \right| = \left| \begin{bmatrix} 1+p_{11}c_1 & p_{12}c_2 \\ p_{21}c_1 & 1+p_{22}c_2 \end{bmatrix} \right|$$

LU decomposition (i.e., Gauss elimination) gives

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[1]Park, M.S., A new approach for multivariable QFT, PhD Thesis, Mech. Eng. Dept, UMass, Feb. 1994.

$$\begin{bmatrix} 1 & 0 \\ -\dfrac{p_{21}c_1}{1+p_{11}c_1} & 1 \end{bmatrix} \begin{bmatrix} 1+p_{11}c_1 & p_{12}c_2 \\ p_{21}c_1 & 1+p_{22}c_2 \end{bmatrix} = \begin{bmatrix} 1+p_{11}c_1 & p_{12}c_2 \\ 0 & 1+p_{22}^2c_2 \end{bmatrix}$$

where

The direct scheme requires stabilization of the plant p_{22}^2. In the inversion-based scheme, the (effective plant to be stabilized is

Recall

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}^{-1} = \frac{1}{\det P} \begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix} \triangleq \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

and

$$\det\left(P^{-1}\right) = 1/_{\det P}$$

so

$$1/\pi_{22}^2 = \frac{\pi_{11}+c_1}{\det\pi + c_1\pi_{22}} = \frac{\dfrac{p_{22}}{\det P}+c_1}{\dfrac{1}{\det P}+c_1\dfrac{p_{11}}{\det P}} = \frac{\det(P)c_1+p_{22}}{1+p_{11}c_1} = p_{22}^2\,!$$

The effective plants at the last design step are equivalent. Note that different c_1 controllers will be designed since plants and performance bounds are different at the 1st step in direct and inverse-based schemes.

General comments. Let the MTF in $Pu = y$ be defined using 2 polynomial matrices Eu=Dy,

$$P = D^{-1}E = \frac{\mathrm{adj}(D)E}{\mathrm{detD}}$$

and

$$P^{-1} = E^{-1}D = \frac{\mathrm{adj}(E)D}{\mathrm{detE}}.$$

Assuming no pole-zero cancellations, the multivariable (transmission) zeros of P are the roots of $\det E$, and the multivariable poles are the roots of $\det D$.

The poles p_{ij}^k of the plants in DS are subset of the MIMO poles. The zeros of the individual plant bear no relation to MIMO zeros. The plant at the final design step is directly related to MIMO poles and zeros (including MIMO nmp zeros).

The poles $1/\pi_{ij}^k$ of the plants in IS are subset of the MIMO poles. The zeros of the individual plant are subset of MIMO zeros. The plant at the final design step is directly related to MIMO poles and zeros (including MIMO nmp zeros).

Consider an example (Murray, 2004).

$$P = \begin{bmatrix} \frac{s^2+5s+10}{(s+1)(s+3)(s+10)} & -\frac{(s+4)(s+5)}{(s+1)(s+3)^2(s+10)} & -\frac{6s^2+29s+44}{(s+1)(s+3)^2(s+10)} \\ \frac{2s+14}{(s+1)(s+3)(s+10)} & \frac{\left(s^2+2s-5\right)(s+5)}{(s+1)(s+3)^2(s+10)} & -\frac{s^2+14s+55}{(s+1)(s+3)^2(s+10)} \\ -\frac{3s-3}{(s+1)(s+3)(s+10)} & \frac{4s^2+13s-10}{(s+1)(s+3)^2(s+10)} & \frac{s^3+5s^2+13s-12}{(s+1)(s+3)^2(s+10)} \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} \frac{\left(s^3+5s^2+7s+15\right)(s+10)}{(s-1)(s+1)(s+6)} & \frac{\left(s^2-17s-12\right)(s+10)}{(s-1)(s+1)(s+6)} & \frac{6s(s+5)(s+10)}{(s-1)(s+1)(s+6)} \\ -\frac{\left(2s^2+13s-1\right)(s+10)}{(s-1)(s+1)(s+6)} & \frac{\left(s^3+6s^2+3s+4\right)(s+10)}{(s-1)(s+1)(s+6)} & \frac{\left(s^2+3s-22\right)(s+10)}{(s-1)(s+1)(s+6)} \\ \frac{(3s+11)(s+10)}{(s-1)(s+6)} & -\frac{(4s+10)(s+10)}{(s-1)(s+6)} & \frac{(s+2)(s+5)(s+10)}{(s-1)(s+6)} \end{bmatrix}$$

$$\det P = \frac{(s-1)(s+6)}{(s+3)^3(s+10)^3} \Rightarrow$$ MIMO zeros are -6 (mp) and 1 (nmp)

MIMO poles are at -1, -3, and -10 (not counting repeated).

Consider closing loops in order from 1st to 3rd. The effective plants at the 1st step are:

$$p_{11}^{1}=\frac{s^2+5s+10}{(s+1)(s+3)(s+10)} \quad \text{and} \quad \frac{1}{\pi_{11}^{1}}=\frac{(s-1)(s+1)(s+6)}{\left(s^3+5s^2+7s+15\right)(s+10)}$$

and we observe the DS plant having MIMO poles and no MIMO zeros, and the IS plant having MIMO zeros and one MIMO pole.

Hence, the 1st loop design in IS does not suffer from MIMO nmp limitation, while the DS does.

Assume $c_1 = 1$ in both schemes, the effective plants at the 2nd, sequential design step are

$$p_{22}^{2}=\frac{(s+1)\left(s^3+13s^2+18s-48\right)}{(s+3)(s+10)\left(s^3+15s^2+48s+40\right)} \quad \text{and} \quad \frac{1}{\pi_{22}^{2}}=\frac{\left(s^4+16s^3+63s^2+84s+144\right)}{\left(s^4+16s^3+69s^2+121s-116\right)(s+10)}.$$

The nmp zero in p_{22}^{2} is not due to MIMO zeros, rather, it is due to the loop design where c_1 affects both zero and pole locations

$$p_{22}^{2}\triangleq\frac{\det(P)\,c_1+p_{22}}{1+p_{11}c_1}.$$

Similarly, the instability in $1/\pi_{22}^2$ is simply due to c_1 destabilizing the 1st loop

$$1/\pi_{22}^2 = \frac{\pi_{11}+c_1}{\det\pi + c_1\pi_{22}} = \frac{\dfrac{\pi_{11}}{\det\pi} + c_1\dfrac{1}{\det\pi}}{1 + c_1\dfrac{\pi_{22}}{\det\pi}}.$$

Also, this plant in mp, hence, at this point the MIMO nmp zero limitation which affected the 1st loop design, is a no-show here.

Next, assume $c_2 = 1$. This choice stabilizes both effective plants. Hence, at the 3rd, and final, design step, the DS effective plant will be stable but must suffer from MIMP nmp zero as shown below

$$p_3^3 = \frac{1}{\pi_{33}^3} = \frac{\left(s^5+27s^4+245s^3+874s^2+1178s-1016\right)}{(s+10)\left(s^5+29s^4+291s^3+1197s^2+2002s+960\right)} \; .$$

In summary, nmp MIMO zeros will pose BW limitation in certain loops, in both schemes. The particular loops to suffer such limitations depends on order of closer, choice of controllers, and nature of the MIMO nmp zero. This is discussed next.

15.3. MIMO NMP LIMITATIONS

As expected, the MIMO case is more complicated[1].

Consider a standard unity negative feedback system with a a MIMO plant P and a diagonal controller C. In what follows, we show how the MIMO nmp zeros affect the sensitivity MTF

$$S = (I + PC)^{-1} \equiv (I + L)^{-1}.$$

Specifically, we show how it is not necessary that all elements of S suffer from the nmp zero limitations. That is, certain input/output relations can be designed without such limitations.

Let

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} P_{11}C_1 & P_{12}C_2 \\ P_{21}C_1 & P_{22}C_2 \end{bmatrix}$$

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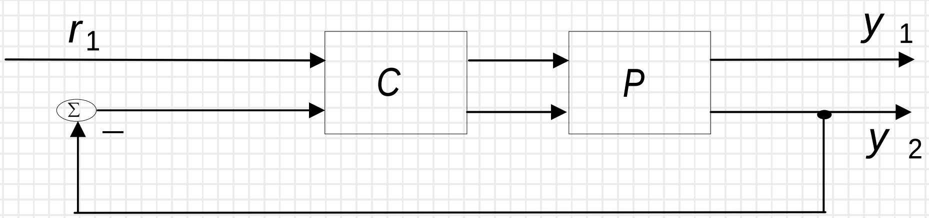
[1]Yaniv O. and Gutman P.O., "Crossover frequency limitations in mimo nonminimum phase feedback systems," *IEEE Trans Automatic Control*, Vol. 47(9), 2002, pg., 1560-1564.

where

$$L_{11} \text{ is a } k \times k \text{ matrix and } C = \text{diag}(C_1, C_2).$$

Now consider the feedback system shown below.



The MTF from r_1 to y_1 is

$$\tilde{L}_{11} = L_{11} - L_{12}(I + L_{22})^{-1}L_{21} = L_1 \begin{bmatrix} I \\ -(I + L_{22})^{-1}L_{21} \end{bmatrix}$$

where

$$L_1 = \begin{bmatrix} L_{11} & L_{12} \end{bmatrix} = \begin{bmatrix} P_{11}C_1 & P_{12}C_2 \end{bmatrix}.$$

Lemma 1. Suppose that

(i) C_1, C_2 and $\tilde{L}_{11}$ are full rank

(ii) any NMP zero of $\begin{bmatrix} P_{11} & P_{12} \end{bmatrix}$ is not a pole of $\begin{bmatrix} P_{11} & P_{12} \end{bmatrix}$ or a pole of L_1 or a pole of $\tilde{L}_{11}$.

Then,

each NMP zero of $\begin{bmatrix} P_{11} & P_{12} \end{bmatrix}$ is an NMP zero of $\tilde{L}_{11}$.

Note that conditions (I) and (ii) are generally satisfied. Next, let

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

be the partition of the closed-loop system corresponding to the above partition. Using the block matrix inversion formula we have

$$S_{11} = (I + \tilde{L}_{11})^{-1}$$

which is the sensitivity MTF from r_1 to y_1.

This means that if the matrix formed from rows $1,\dots,m$ of P have NMP zeros, then at least one of the SISO sensitivity functions s_{ii}, $i = 1,\dots,m$, must suffer the NMP zero limitations (i.e., limited crossover frequency).

Note that the above is true for any set of rows in P, not necessarily in order. This leads to the key result.

Theorem. Consider the above partitioned feedback system. Assume that it is closed-loop stable, that L_{11} is NMP, and that the conditions of the Lemma are satisfied. Then at least one of the actual loop transmissions

$$c_i p_{ii}^n,\ i = 1,\dots,n,$$

must suffer from crossover limitations related to the NMP zeros of

$$\begin{bmatrix} P_{11} & P_{12} \end{bmatrix}$$

where P_{11} and P_{21} each have m rows.

Remarks:

- If det$P$ has NMP zeros, but no combinations of rows of P drop rank (other then all of them), then we can assign crossover limitation to any s_{ii}.
- Say we have a 4x4 plant. Then if rows {1,2} drop rank and rows {3,4} also drop rank, than at least one s_{ii} from rows {1,2} and one s_{ii} from rows {3,4} must suffer from crossover limitations.
- If some combinations of rows also have NMP zeros, it is possible, in general, to select the rows of S that will suffer the crossover limitations.
- Increasing the number of plant inputs may remove a MIMO NMP zero.

Example: Consider the plant

$$P = \frac{1}{s+1}\begin{bmatrix} 1 & .1-s \\ -.1 & 1 \end{bmatrix}$$

It has an NMP zeros at 10.1 (`tzero(P)`). None of its rows has any finite zeros. Also

$$\det P = \frac{-0.1s + 1.01}{(s+1)^2}$$

with

$$P^{-1} = \begin{bmatrix} \dfrac{s+1}{-0.1s+1.01} & \dfrac{0.1(s+1)}{-} \\ \dfrac{0.1(s+1)}{-0.1s+1.01} & \dfrac{s+1}{-0.1s+1.01} \end{bmatrix} \equiv \left[\pi_{ij}\right]$$

We observe that both open-loop plants $1/\pi_{11}$ and $1/\pi_{22}$ are NMP due to det$P$. If we design using direct procedure, neither p_{11} nor p_{22} are NMP; hence, we would have to design loop 2 1st if we wanted to assign NMP limitation to 1st loop.

Arbitrarily select to have the 1st loop suffer crossover limitations. The choice of c_1 = 1 stabilizes this loop (setting aside performance considerations) since

$$1+\frac{c_1}{\pi_{11}} = 1+\frac{-.1s+1.01}{s+1} = \frac{.9s+2.01}{s+1}.$$

The effective plant in the 2nd (and last) loop is

$$\pi_{22}^2 = \pi_{22} - \frac{\pi_{12}\pi_{21}}{c_1+\pi_{11}} = \frac{-0.0900s^3 + 0.6080s^2 + 2.8291s + 2.1311}{0.9\,s^3 + 3.81\,s^2 + 4.92s + 2.01}$$

$$\frac{1}{\pi_{22}^2} = \frac{0.9\,s^3 + 3.81\,s^2 + 4.92s + 2.01}{-0.0900s^3 + 0.6080s^2 + 2.8291s + 2.1311}$$

so the effective plant is minimum-phase (but unstable).

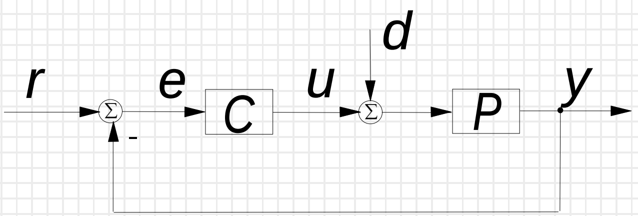
Can we design c_1 such that $1/\pi_{22}^2$ is also mp?

$$\frac{1}{\pi_{22}^2} = \begin{cases} \dfrac{\dfrac{\pi_{11}}{\det\pi} + c_1 \dfrac{1}{\det\pi}}{1 + c_1 \dfrac{\pi_{22}}{\det\pi}} \\ \dfrac{1 + c_1 p_{22}}{\dfrac{\det\pi}{\pi_{11}} + c_1 \dfrac{\pi_{22}}{\pi_{11}}} \end{cases}$$

In summary, in order to make future design steps free of nmp zeros and/or unstable poles, present design must not only stabilize its effective plant, but also additional plant as shown above. Yes, MIMO is complicated....

15.4. Design algorithms for SISO Elements

Consider again the design problem posed in Chap. 14. The block diagram is shown below.



$$T = (I + PC)^{-1} P = \left[t_{ij} \right].$$

The control problem involves the design of an LTI *nxn* diagonal controller *C* that achieves:

- Robust stability, and
- $\left| t_{ij}(j\omega) \right| \le \alpha_{ij}(\omega), \quad i, j = 1, \ldots n, \forall P \in \mathcal{P}.$

Here we are enforcing specific amplitude constraint on each siso TF in contrast to the input/output constraint used in Chap. 14. This specific problem formulation is used more often.

The design algorithms can be readily derived from the ones in Chap. 14. If the inputs are impulses, then the outputs, impulse responses, are also siso elements of the closed-loop MTF.

Specifically,

$$y=(I+PC)^{-1}Pd=\begin{bmatrix}1+p_{11}c_1 & p_{12}c_2\\ p_{21}c_1 & 1+p_{22}c_2\end{bmatrix}^{-1}\begin{bmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix}\begin{bmatrix}d_1 & 0\\ 0 & d_2\end{bmatrix}$$

$$=\begin{bmatrix}t_{11} & t_{12}\\ t_{21} & t_{22}\end{bmatrix}\begin{bmatrix}d_1 & 0\\ 0 & d_2\end{bmatrix}$$

and when the inputs are impulses, d_1 = 1 and d_2 = 2, and they appear only one at a time (one is on and the other off) we have

$$t_{11}=y_1(d_1=1,d_2=0)$$

$$t_{12}=y_1(d_1=0,d_2=1)$$

$$t_{21}=y_2(d_1=1,d_2=0)$$

$$t_{22}=y_2(d_1=0,d_2=0)$$

and the design algorithms for computing bounds are precisely those derived in Chap. 14:

$$(P^{-1}+C)y=d \Rightarrow \begin{bmatrix} \pi_{11}+c_1 & \pi_{12} \\ \pi_{21} & \pi_{22}+c_2 \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where each output y_{ij} correspond to a specific input configuration. And straightforward adaptation of the algorithms in Chap. 14 gives

$$|t_{11}| = |y_{11}| = \left|\frac{d_1-\pi_{12}y_{21}}{\pi_{11}+c_1}\right| = \left|\frac{1-\pi_{12}t_{21}}{\pi_{11}+c_1}\right| \geq \left|\frac{1+|\pi_{12}|\alpha_{21}}{\pi_{11}+c_1}\right|$$

$$|t_{12}| = |y_{12}| = \left|\frac{-\pi_{12}y_{22}}{\pi_{11}+c_1}\right| = \left|\frac{-\pi_{12}t_{22}}{\pi_{11}+c_1}\right| \geq \left|\frac{|\pi_{12}|\alpha_{22}}{\pi_{11}+c_1}\right|.$$

At the 2nd design step, the algorithms are

$$\begin{bmatrix} \pi_{11}+c_1 & \pi_{12} \\ 0 & \pi_{22}^2+c_2 \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{-\pi_{21}}{\pi_{11}+c_1} & 1 \end{bmatrix}$$

$$|t_{21}| = |y_{21}| = \left|\frac{\frac{-\pi_{21}}{\pi_{11}+c_1}}{\pi_{22}^2+c_2}\right| \leq \alpha_{21}$$

$$|t_{22}| = |y_{22}| = \left|\frac{1}{\pi_{22}^2+c_2}\right| \leq \alpha_{22}.$$

Recall that the siso elements relates the inputs and output as follows

$$y_1 = t_{11}d_1 + t_{12}d_2$$
$$y_2 = t_{21}d_1 + t_{22}d_2$$

and we conclude that using output constraint formulation takes into account not only the individual siso TFs but also the nature of the inputs.

15.5. Direct Design algorithms

Consider again the design problem from 15.4.

$$T = \begin{bmatrix} 1+p_{11}c_1 & p_{12}c_2 \\ p_{21}c_1 & 1+p_{22}c_2 \end{bmatrix}^{-1} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$\begin{bmatrix} 1+p_{11}c_1 & p_{12}c_2 \\ p_{21}c_1 & 1+p_{22}c_2 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}.$$

Isolating the 1st loop

$$\begin{bmatrix} 1+p_{11}c_1 & p_{12}c_2 \\ p_{21}c_1 & 1+p_{22}c_2 \end{bmatrix} \begin{bmatrix} t_{11} \\ t_{21} \end{bmatrix} = \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix}$$

interchanging elements

leads to

$$t_{11} = \frac{\left|\begin{bmatrix} p_{11} & p_{12} \\ p_{21} - t_{21} & p_{22} \end{bmatrix}\right|}{\left|\begin{bmatrix} 1 + p_{11}c_1 & p_{12} \\ p_{21}c_1 & p_{22} \end{bmatrix}\right|} = \frac{\det P + p_{12}t_{21}}{1 + c_1 \det P}$$

and the related (conservative) inequality for bound computation

$$|t_{11}| = \left|\frac{\det P + p_{12}t_{21}}{1 + c_1 \det P}\right|$$

Similarly, we can derive a design inequality for t_{12}.

Once c_1 is designed, we use Gauss elimination into the working matrices. We end up with design inequalities that do not require overdesign.

15.6. Homework

1. Consider the uncertain plant family

$$\mathcal{P} = \left\{ \begin{array}{l} P(s) = \frac{1}{s^2}\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} : \\ k_{11} \in [2,5],\ k_{22} \in [2,4],\ k_{12} \in [-.5,.5],\ k_{21} \in [.3,1.5] \end{array} \right\}$$

and plant output disturbance configurations

$$|d_1(j\omega)| = \frac{1}{\omega} \quad \text{and} \quad d_2 = 0, \ \text{ or}$$

$$|d_2(j\omega)| = \frac{1}{\omega} \quad \text{and} \quad d_1 = 0.$$

Design a diagonal controller C such that the closed-loop system achieves

- Robust stability, and
- $|y_i(j\omega)| \le \alpha_i(\omega), \quad i = 1,2, \ \forall P \in \mathcal{P}$ and $d \in \mathcal{d}$.

where

ω	1	2	3
α_1	-26 dB	-20	-14
α_2	-24	-18	-16

You should investigate the stability and nmp/mp nature of the effective plant at the last step.

The choice of loop closure is yours, however, you must tune your design to minimize the overdesign inherent in the 1st step.

Show all relevant work (avoid figures w/o an explanation).