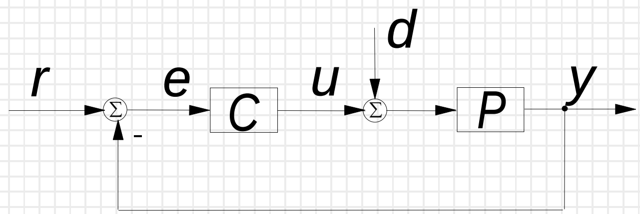
14. Multivariable QFT Design (Part 1)

In this chapter we begin our introduction to multivariable (MIMO) QFT design for feedback systems such as shown below. We start by focusing on square plants, i.e., plants with same number of inputs as outputs. We also assume that P has an inverse.



We first present the design procedure constructively for a plant input disturbance rejection problem, then deal with some theoretical issues and close with an illustrative example. Assume the uncertain plant belongs to a set

$$P \in \mathcal{P}, \quad P \text{ is an } n \times n \text{ LTI MTF}$$

and that the disturbance belongs to a set

$$d \in \mathcal{d}.$$

The control problem involves the design of an LTI *nxn* diagonal controller *C* that achieves:

where $y = [y_1, y_2, \ldots, y_n]'$.

The design procedure for a 2x2 plant is developed in the following way.

Substituting in the notation

we get

where $d = [d_1, d_2]'$.

We would like to separately (sequentially) design the two SISO controllers, c_1 and c_2 which allows two SISO QFT designs. This can be accomplished using Gauss elimination. Specifically, multiplying both sides on the left with

gives

where

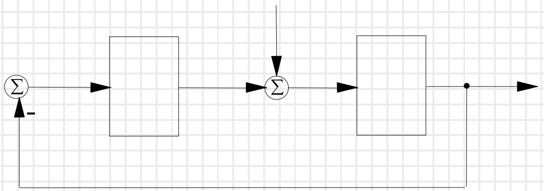
We can now isolate the outputs

The design procedure is sequential. In this exposition, we start with the design of c_1 such that

One difficulty here is that y_2 is unknown. On one hand (setting aside stability considerations), high gain feedback would do the job assuming y_2 is bounded. But we already saw in earlier chapters the disadvantages of high loop gains beyond what is really needed.

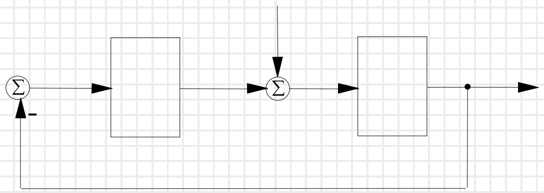
To get around this problem we assume the MIMO design can be successfully completed. Invoking the triangle inequality and a worst case response gives a conservative constraint

The above has a single-loop interpretation as a plant input disturbance rejection problem where the MIMO interaction has been lumped together with the actual disturbances. This is illustrated in the block diagram below.



If a SISO controller c_1 can be designed to robustly stabilize the plant $1/\pi_{11}$ and satisfy the spec on $|y_1|$, then we can proceed to the next step in the sequential procedure.

Specifically, since c_1 is known, the 2nd step is also an input disturbance rejection problem as shown below



Unlike the situation in the previous step, no assumption is needed and hence the design is exact in terms of a single unknown controller c_2:

If a SISO controller c_2 can be designed to robustly stabilize the plant $1/\pi^2_{22}$ and satisfy the spec on $|y_2|$ we are done.

Before we go ahead with a design example, let us discuss the multivariable stability problem.

14.1 Stability Considerations

To achieve closed-loop stability, we need to worry about two special cases (though you are unlikely to see one in your applications). Assume that det(I+PC) is not identically zero and that P and C are at least proper. A fixed unstable decentralized mode is a scenario where you cannot stabilize a multivariable plant with a diagonal controller.

For example, the unstable plant

$$P = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s-1} \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

is not stabilized by this diagonal controller

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

even though the diagonal controllers stabilize their corresponding loops with both siso characteristic equations stable

The sensitivity is

This is not due the sequential procedure. The lack of plant interaction does not allow stabilization via this diagonal controller.

How to identify fixed decentralized modes (Davison, 1976): A minor modification here is the assumption of controllable and observable system.

1. Obtain a state-space description (A,B,C): $P(s) = C(sI-A)^{-1}B$.
2. Find the eigenvalues of the open-loop system $\lambda(A)$.
3. Choose and arbitrary diagonal static controller K and scale it such that $\|A\| \approx \|BKC\|$.
4. Find $\lambda(A+BKC)$; those equal to the open-loop eigenvalues are possible hidden modes.
5. Repeat steps 3-4 until all the fixed modes of A are identified.

Davison, E.J., 1976, Decentralized stabilization and regulation in large multivariable systems, in Ho & Mitter, Eds., *Direction in large Scale Systems*, pp. 303-323, Plenum Press, NY.

Another special case occur when there is an unstable pole/zero cancellation when forming *PC* or *CP*. Unlike SISO plants where such cancellation is readily visible, the multivariable case is more complex as seen in the following example. Consider the plant

$$P = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s+1} \\ 0 & \frac{1}{s-3} \end{bmatrix}$$

and the diagonal controller

$$C = \begin{bmatrix} \frac{s-1}{s+2} & 0 \\ 0 & \frac{2(s-3)}{s+2} \end{bmatrix}$$

which was designed to "cancel" unstable poles since

$$PC = \begin{bmatrix} \frac{1}{s+2} & \frac{2(s-3)}{(s+1)(s+2)} \\ 0 & \frac{2}{s+2} \end{bmatrix}$$

and

$$(I+PC)^{-1} = \begin{bmatrix} \frac{s+2}{s+4} & \frac{-2(s+2)}{(s+1)(s+4)} \\ 0 & \frac{s+2}{s+3} \end{bmatrix}.$$

Not so fast. This closed-loop MTF is not internally stable since

This instability is due to multivariable pole-zero cancellation. While not minimizing importance of *internal stability*, exact cancellations are simply not going to occur in when the controller is designed manually. And the related considerations that must be included in theory are less critical for us.

Hence, in what follows we have the standing assumptions that our system does not have unstable decentralized fixed modes, does not have unstable multivariable pole-zero cancellations and that $\det(I+PC)\neq 0$.

Theorem. Consider a 2x2 MTF P and a diagonal controller C. Assume there are no fixed, unstable decentralized modes and no unstable pole cancellations between P and C. Then the closed-loop system is stable iff c_2 stabilizes $1/\pi^2_{22}$.

In addition, closed-loop stability can also be inferred when using non-inversion based algorithms. In this case the Nyquist generalized criterion is applied to the plot of det(I+PC). Here, we also used a sequential procedure. Assume that the 1st loop is designed first. Under the same assumption as above, the closed-loop system is stable iff c_2 stabilizes p^2_{22} .

Proof. This is HW problem. Prove for both inversion and non-inversion procedures. Also show the relation between $1/\pi^2_{22}$ and p^2_{22}. Show details on direct design.

We'll return to stability issues later.

14.2. A 2x2 Example

Consider the uncertain plant family

$$\mathcal{P} = \left\{ \begin{array}{l} P(s) = \frac{1}{s}\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} : \\ k_{11} \in [2,4],\ k_{22} \in [2,4],\ k_{12} \in [1,1.8],\ k_{21} \in [1,1.8] \end{array} \right\}$$

and plant input disturbance configurations

$$|d_1(j\omega)| = \tfrac{1}{\omega} \quad \text{and} \quad d_2 = 0, \quad \text{or}$$

$$|d_2(j\omega)| = \tfrac{1}{\omega} \quad \text{and} \quad d_1 = 0.$$

The closed-loop system should achieve

where

ω	1	2	3
α_1	0.0501	0.0316	0.0251
α_2	0.0501	0.0316	0.0251

We start the sequential procedure by designing the first loop (we'll discuss the order of closure in a later chapter). Taking plant inverse

$$P^{-1} = \frac{s}{k_{11}k_{22} - k_{12}k_{21}} \begin{bmatrix} k_{22} & -k_{12} \\ -k_{21} & k_{11} \end{bmatrix} = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

the SISO plant in the first step is

Performance bounds are computed from the relation derived earlier

$$|y_1(j\omega)| = \left| \frac{d_1 - \pi_{12} y_2}{\pi_{11} + c_1}(j\omega) \right| \le \frac{|d_1| + |\pi_{12}|\,\alpha_2}{|\pi_{11} + c_1|} \le \alpha_1(\omega), \text{ for all } P \in \mathcal{P} \text{ and } d \in \mathcal{d}.$$

For the two disturbance configurations we have

These constrain the loop's response at the low frequency range. As in the SISO case, we should always work with reasonable stability margins applicable at all frequencies. In this example let us use

$$|1 + c_1 / \pi_{11}| \geq 0.6, \quad \text{for all } P \in \mathcal{P}, \omega \geq 0.$$

Recall that these bounds are used as guides for shaping the nominal loop. To achieve robust stability, we must work with the same nominal plant throughout the design procedure. An arbitrary choice of the nominal plant is

$$k_{11} = 2,\ k_{22} = 2,\ k_{12} = 1.8, \text{ and } k_{21} = 1.8,$$

for which

The inequalities for computing bounds here may not have the forms seen in the SISO feedback problems of earlier chapters. However, just as we did there, bounds are actually computed to constrain the controller's response. We convert them to bounds on a nominal loop to achieve stability. The computations are done using a toolbox function for computing bounds in multivariable problems.

There are two generic inequalities that capture multivariable problems. These are

ptype	I/O Problem
10	$\left\lvert \frac{A+Bg}{C+Dg} \right\rvert \le Ws_{10}$
11	$\left\lvert \frac{\lvert A \rvert + \lvert Bg \rvert}{C+Dg} \right\rvert \le Ws_{11}$

```
bdb = genbnds(ptype,w,Ws,A,B,C,D,Pnom)
```


The first problem type is the standard linear fractional transformation which captures all nine SISO problems covered earlier. The second type fits most problems encountered in intermediate steps of a multivariable QFT design. For example,

$$\left|\frac{\frac{1}{\omega}+|\pi_{12}|\alpha_2}{\pi_{11}+c_1}\right| \quad \Rightarrow$$

$$\left|\frac{|\pi_{12}|\alpha_2}{\pi_{11}+c_1}\right| \quad \Rightarrow$$

We are now ready to design.

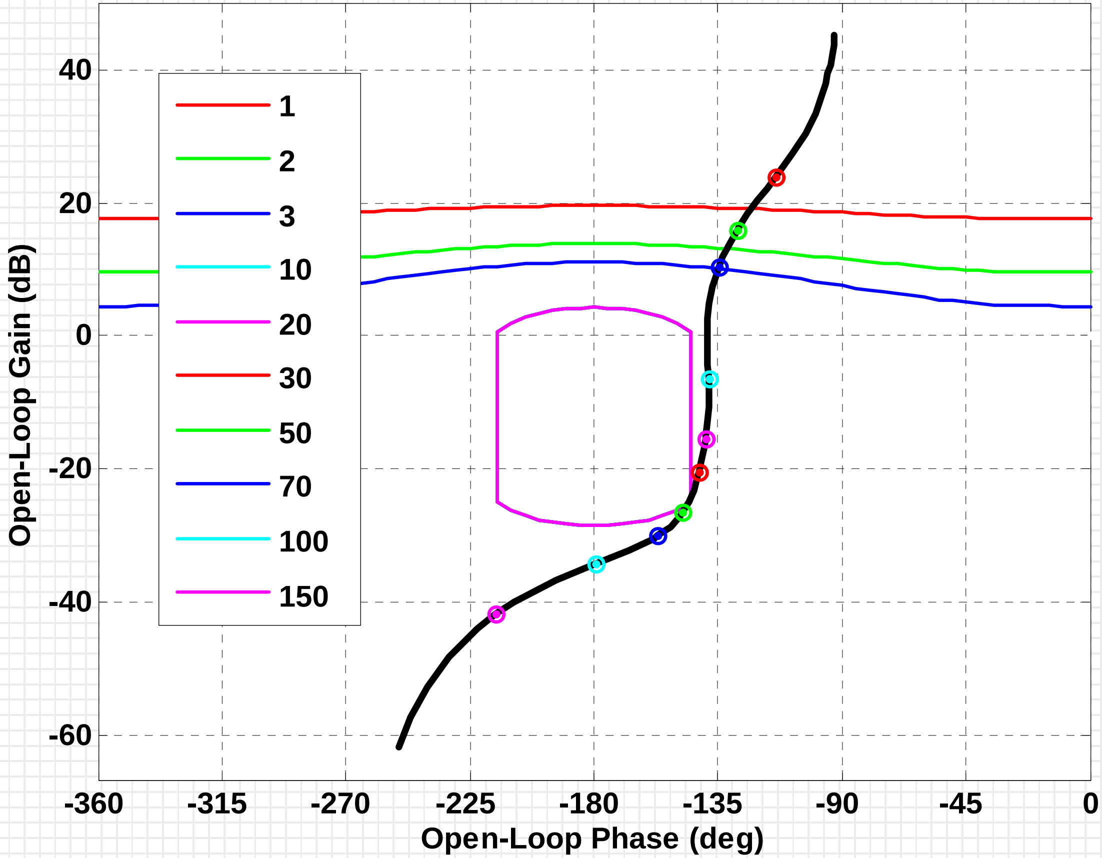
The template of the uncertain plant in the first step

$$\frac{1}{\pi_{11}} = \frac{(k_{11}k_{22} - k_{12}k_{12})/k_{22}}{s}$$

has only gain uncertainty and will be the vertical line on a NC. All details of bound computation and loop shaping can be found in `ch11_ex1_loop1.m`.

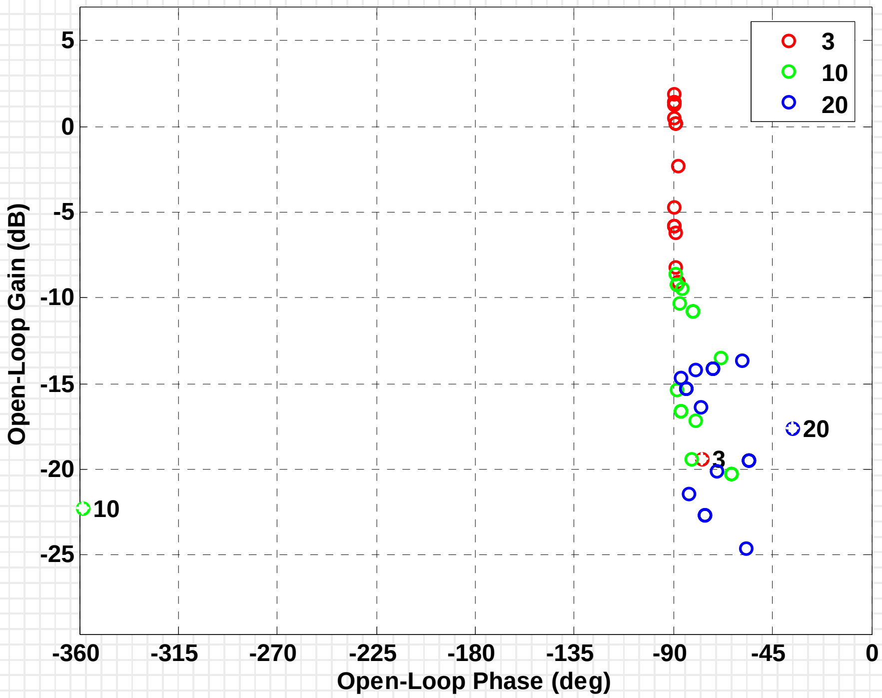
Let us take a look at the M file.

It is clear from inspection of the numerators in the performance inequalities that the first one is “tougher”. So we need not compute bounds for the second one. The computed bounds and the designed nominal loop are shown below.



We proceed to the second (and last) step in the sequential procedure. The plant used in this step takes into account that the other loop is already designed (i.e., closed)

Unlike the plant set in the first step, this one is more complex due to the multivariable interaction with the controller. We should expect its templates to exhibit more than a simple gain variations. This is depicted below for some "mid-range" frequencies.



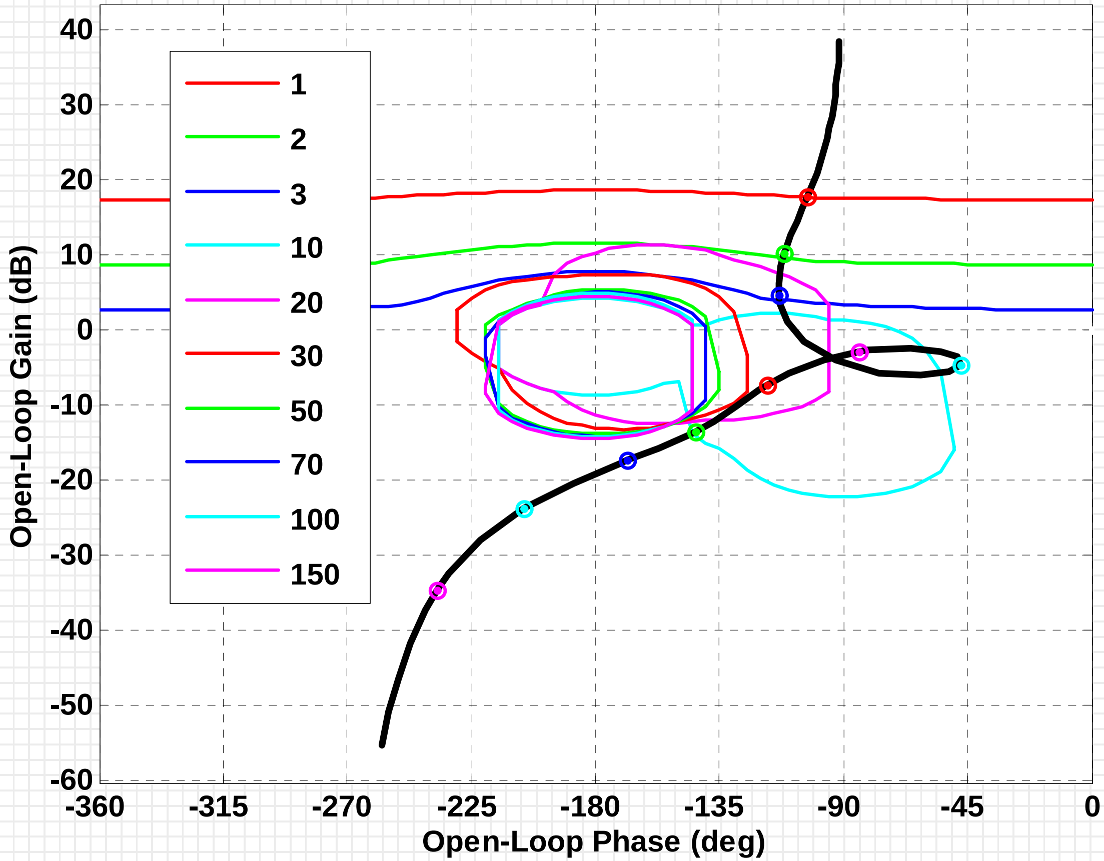
The nominal plant transfer function also shows this complex nature

$$\frac{1}{\pi_{22}^2} = \frac{0.38(s/18.2+1)(s^2/6^2+0.8s/6+1)(s^2/119^2+1.1s/119+1)}{s(s/11.5+1)(s^2/19^2+0.9s/19+1)(s^2/115^2+1.1s/115+1)}.$$

Two sets of bounds are computed for the two disturbance configurations

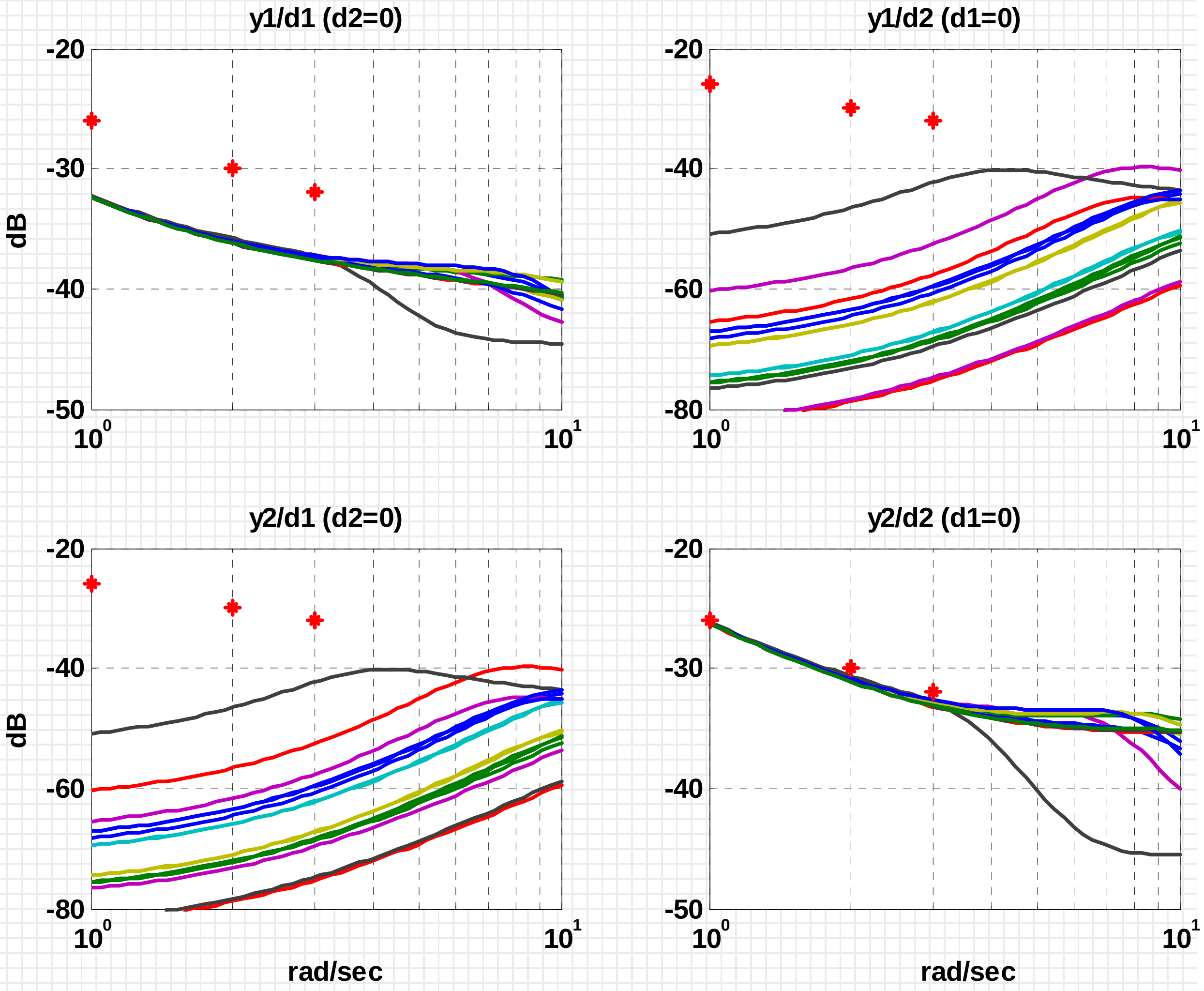
along with margin bounds

The computed bounds and the designed nominal loop are shown below.



We are almost done. Evaluation of the closed-loop responses completes the design.

A comparison of closed-loop frequency response sets from *d* to *y* and the performance weights is shown below.



Based on what we have learned so far, we can characterize the MIMO QFT design procedure for a 2x2 system by the following:

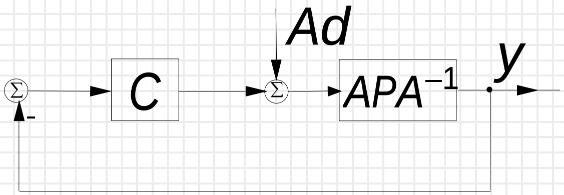
14.3. Designing the 2nd Loop First

The technique developed earlier assumed that the first loop is designed first. In general, we should not automatically assume this is true. For comparison purposes, let us now consider the same problem but design the 2nd loop first.

There are two approaches. We can re-derive the algorithms by explicitly starting with design of c_1 to meet the specification on y_2. However, an easier method is to permute the MTFs which allows us to maintain the same algorithms and notation. This is done as follows.

Let A [*Yaniv*, Ch. 4] be an nxn row or column permutation matrix reflecting the order of loop design. For example, in a 2x2 system, designing the 2nd loop first implies that

Suppose that the diagonal C robustly stabilizes the system shown below



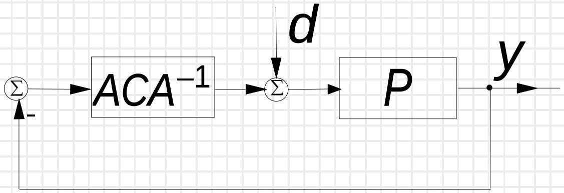
and achieves the performance

$$\begin{vmatrix} y_1(j\omega) \\ y_2(j\omega) \end{vmatrix} \leq A \begin{bmatrix} \alpha_1(\omega) \\ \alpha_2(\omega) \end{bmatrix},$$

then the controller

$$A^{-1}CA$$

also solves the previous problem (where the 1st loop was designed first):



The proof is straightforward. From the block diagram

After some basic matrix algebra we get

$$
\begin{aligned}
y &= (I + APA^{-1}C)^{-1}APd \\
&= \left[A^{-1}(I + APA^{-1}C)\right]^{-1} Pd \\
&= (A^{-1} + PA^{-1}C)^{-1}Pd \\
&= \left[(I + APA^{-1}CA)A^{-1}\right]^{-1} Pd \\
&= A(I + PA^{-1}CA)^{-1}Pd.
\end{aligned}
$$

Comparing that with the output in the original block diagram without permutations (call it y_o and the controller is C_o)

we see that

$$y = Ay_o.$$

Recall that the original specifications on y were also pre-multiplied by A. Since A can only permute, we conclude that if

then

$$|y(j\omega)| = |Ay_O(j\omega)| \le A\begin{bmatrix}\alpha_1(\omega)\\ \alpha_2(\omega)\end{bmatrix},\ k = 1,\ldots n,\ \forall P \in \mathcal{P} \text{ and } \forall d \in \mathcal{d}$$

$$\Rightarrow \quad |y_O(j\omega)| \le \begin{bmatrix}\alpha_1(\omega)\\ \alpha_2(\omega)\end{bmatrix}, \quad k = 1,\ldots n,\ \forall P \in \mathcal{P} \text{ and } \forall d \in \mathcal{d}.$$

In our 2x2 system, a design starting with the 2nd loop would result in the following controller for the original (non-permuted) system

$$C_O = A^{-1}CA = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}^{-1}\begin{bmatrix}c_1 & 0\\ 0 & c_2\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} =$$

Back to our example. To design the 2nd loop first, we permute the plant

$$APA^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} p_{22} & p_{21} \\ p_{12} & p_{11} \end{bmatrix}^{-1} = \begin{bmatrix} \pi_{22} & \pi_{21} \\ \pi_{12} & \pi_{11} \end{bmatrix}.$$

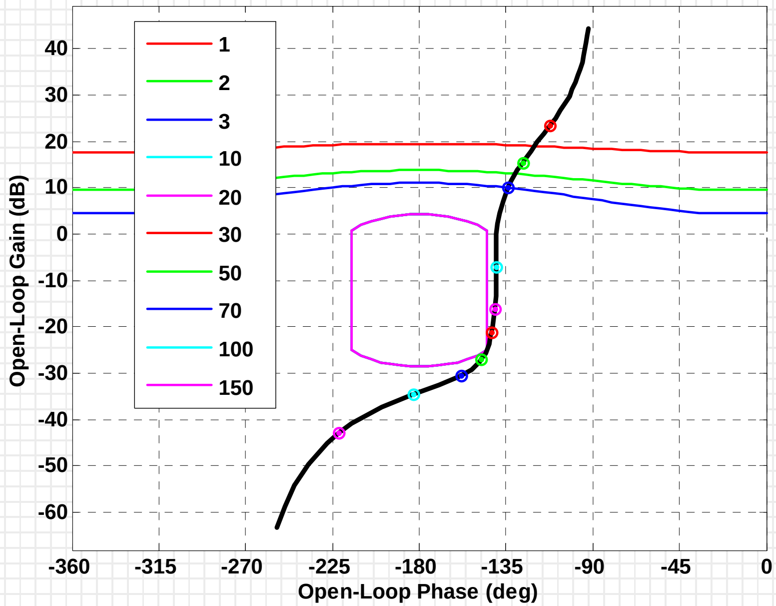
The permuted disturbance and specification vectors are

$$Ad = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} =$$

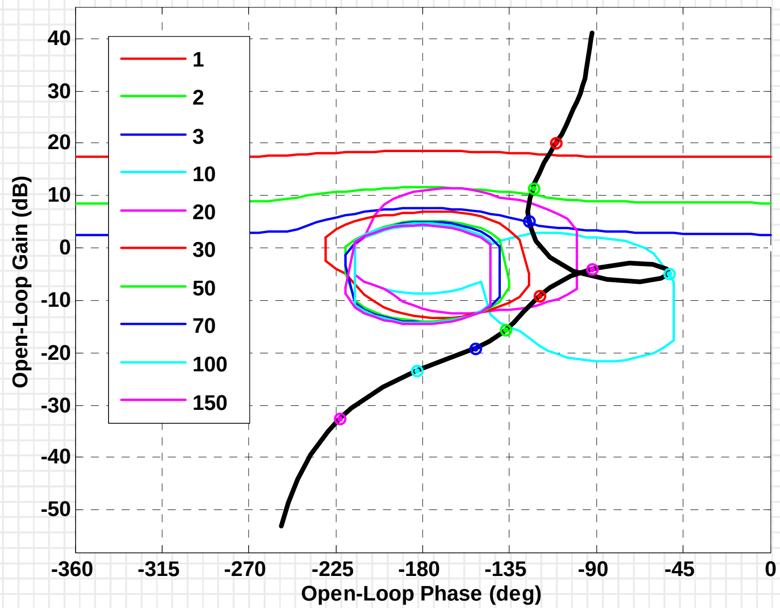
$$A \begin{bmatrix} \alpha_1(\omega) \\ \alpha_2(\omega) \end{bmatrix} =$$

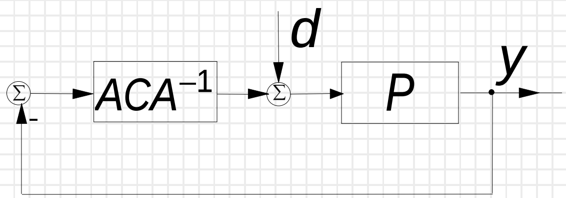
Following the same sequential design algorithms using in the previous example, we design c_2 first then c_1. The details can be found in `ch11_ex1_loop2.m`. The designed nominal loops (after permutation) and closed-loop responses vs. weights are shown.

2nd loop designed first

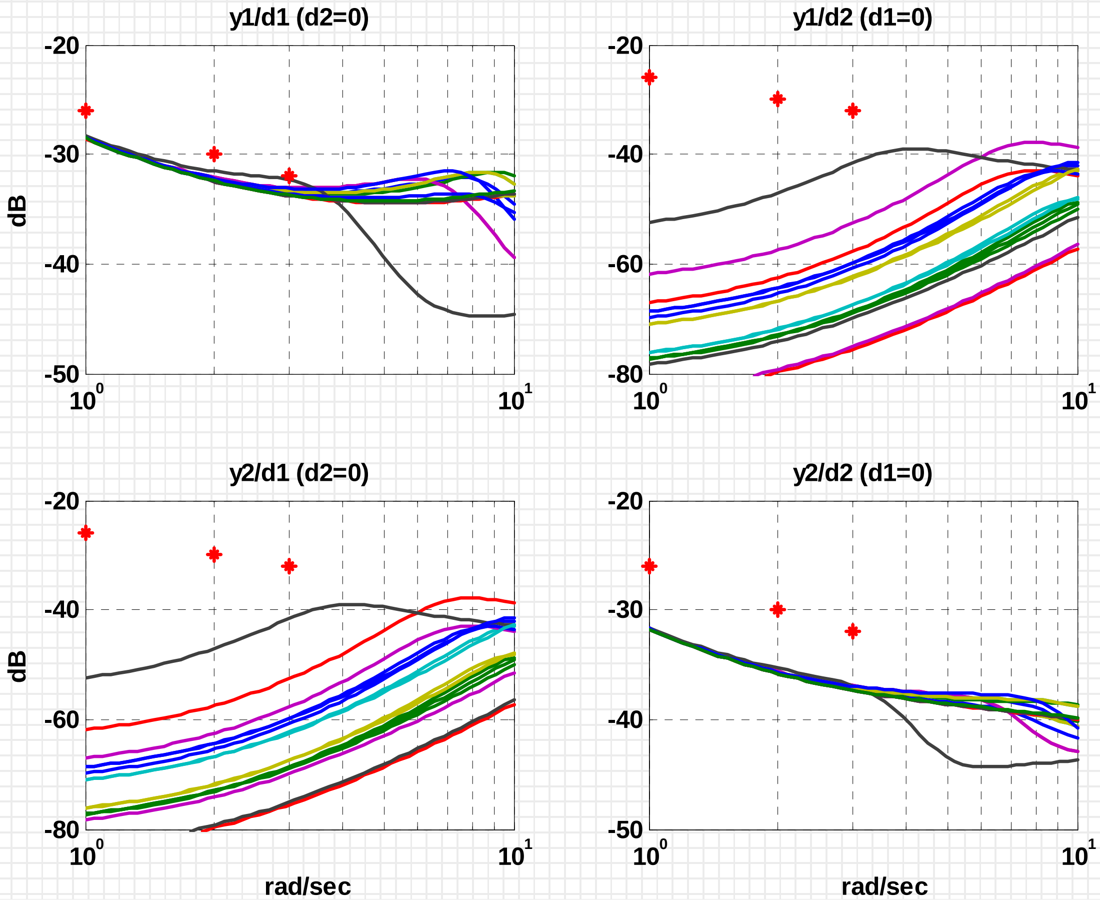


1st loop designed second

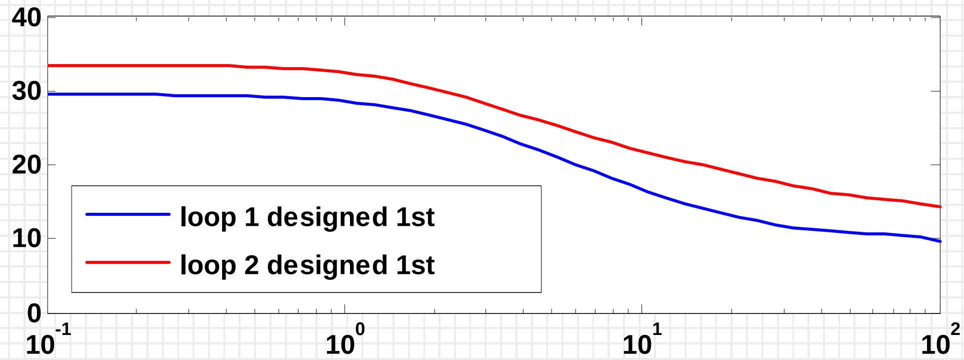


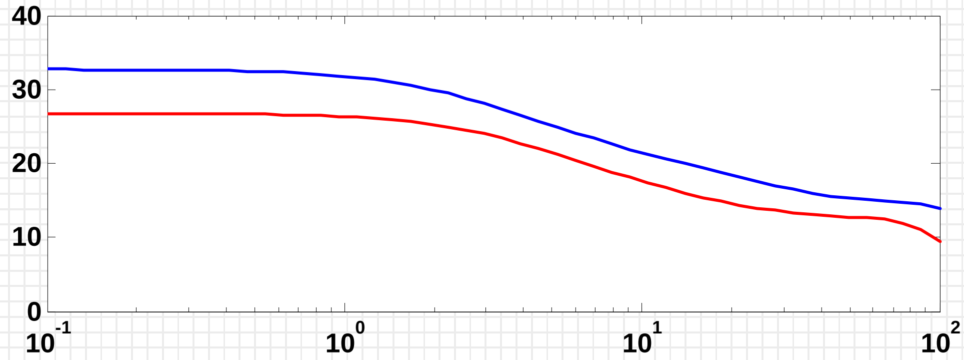


Evaluation of this design is shown below. We observe that the second loop now appears to have the over-design (why?).



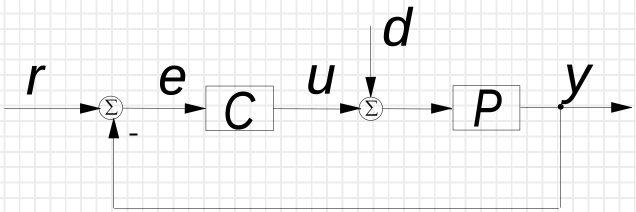
Comparison of the controllers reveals this behavior. In any particular problem, considerations of sensor noise and actuator nonlinearities must be taken into account in deciding the order of loop design. We will see later how non-minimum phase zeros can also affect this choice.





14.4. A General Plant Input Disturbance Rejection Problem

Consider again the plant input disturbance rejection problem in the previous chapter but now assume the plant to have *n* inputs and *n* outputs.



Using the notation

$$P = [p_{ij}],\ P^{-1} = \left[\pi_{ij}^{1}\right]$$

$$C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$$

$$d = [d_1, d_2, \ldots, d_n]^T\ ,$$

$$y = [y_1, y_2, \ldots, y_n]^T\ .$$

The robust performance problem is:

- Robust stability, and
- $|y_k(j\omega)| \le \alpha_k(\omega), \quad k = 1, \ldots n$, for all $P \in \mathcal{P}$ and $d \in \mathcal{d}$.

We perform the standard operations used in the 2x2 case:

$$\begin{aligned} y =& \ (I+PC)^{-1}Pd \\ (I+PC)y =& \ Pd \\ (P^{-1}+C)\mathrm{y} =& \ d. \end{aligned}$$

And using the above notation we have in detail

$$\begin{bmatrix} \pi_{11}^1 + c_1 & \pi_{12}^1 & \cdots & \pi_{1n}^1 \\ \pi_{21}^1 & \pi_{22}^1 + c_2 & \cdots & \pi_{2n}^1 \\ \vdots & \vdots & \vdots & \vdots \\ \pi_{n1}^1 & \pi_{n2}^1 & \cdots & \pi_{nn}^1 + c_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} d_1^1 \\ d_n^1 \\ \vdots \\ d_n^1 \end{bmatrix}$$

where the superscript 1 denotes the original plant inverse.

Without loss of generality, we design starting with the first loop in order to the n'th. Isolating the first output gives

$$y_1 = \frac{d_1^1 - \sum_{i=2}^{n} \pi_{1i}^1 y_i}{\pi_{11}^1 + c_1}$$

Spec: $\left| y_1(j\omega) \right| \le \alpha_1(\omega).$

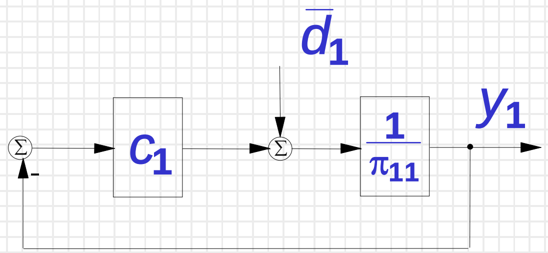
At this stage, the remaining loops have not been designed yet. Hence, we must assume that there exists a controller C that solves the feedback problem. Using triangle inequality

$$\left|y_1(j\omega)\right| = \frac{\left|d_1^1 - \sum_{i=2}^{n} \pi_{1i}^1 y_i\right|}{\left|\pi_{11}^1 + c_1\right|}$$

This above must be satisfied in spite of plant uncertainty, leading to the final inequality used to compute QFT bounds

$$\frac{\left|d_1^1\right| + \sum_{i=2}^{n} \left|\pi_{1i}^1\right| \alpha_i}{\left|\pi_{11}^1 + c_1\right|} \le \alpha_1(j\omega), \quad \forall P \in \mathcal{P}, \ \forall d \in \mathcal{d}.$$

Because of the single unknown term above, we can interpret it as a single-loop feedback problem where the objective is to design c_1 to satisfy the above performance inequality and robustly stabilize the system shown below.



$$\bar{d}_1^1 \equiv |d_1| + \sum_{i=2}^{n} |\pi_{1i}^1| \alpha_i$$

Having successfully designed the first loop, we proceed to the next, second loop. However, to reduce the required upper bounding, we "roll" the known c_1 into the remaining loops. This is done by pre- multiplying both sides of the governing matrix equation with (i.e., Gauss elimination)

$$V_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \frac{-\pi_{21}^1}{\pi_{11}^1 + c_1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{-\pi_{n1}^1}{\pi_{11}^1 + c_1} & 0 & \cdots & 1 \end{bmatrix}$$

giving

$$\begin{bmatrix} \pi_{11}^1 + c_1 & \pi_{12}^1 & \cdots & \pi_{1n}^1 \\ 0 & \pi_{22}^2 + c_2 & \cdots & \pi_{2n}^2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \pi_{n2}^2 & \cdots & \pi_{nn}^2 + c_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} d_1^1 \\ d_2^2 \\ \vdots \\ d_n^2 \end{bmatrix}$$

where

$$\pi_{ij}^2 \equiv \pi_{ij}^1 - \frac{\pi_{i1}^1 \pi_{1j}^1}{\pi_{11}^1 + c_1}, \quad i = 2, \ldots, n, \ j = i+1, \ldots, n,$$

$$d_i^2 \equiv d_i^1 - \frac{\pi_{i1}^1 d_i^1}{\pi_{11}^1 + c_1}, \quad i = 2, \ldots, n.$$

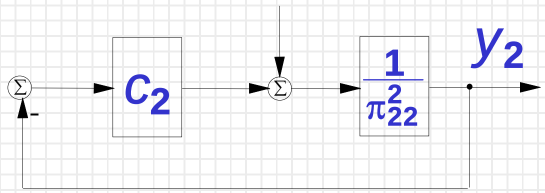
Isolating the second output gives

Note that the equation shows explicit MIMO interaction between the second and 3rd to n'th loops. The interaction with the first loop is made implicit by design via the terms π_{ij}^2 and $\bar{d}_2^2$.

The interaction with the yet to be designed 3rd to n'th loops is replaced using the upper bounding as before resulting in

$$\left|y_2(j\omega)\right| \le \frac{\left|d_2^2\right| + \sum_{i=3}^{n}\left|\pi_{2i}^2\right|\alpha_i}{\left|\pi_{22}^2 + c_2\right|} \le \alpha_2(j\omega), \quad \forall P \in \mathcal{P}, \ \forall d \in \mathcal{d}.$$

This single-loop feedback problem involves design of c_2 to satisfy the above performance inequality and robustly stabilize the system shown below.



If we can successfully complete the design of this loop, we proceed to the next one.

Once again, we “roll” this design into the remaining loops using the pre-multiplying matrix

$$V_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{-\pi_{32}^2}{\pi_{22}^2+c_2} & 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{-\pi_{42}^2}{\pi_{22}^2+c_2} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & \frac{-\pi_{n2}^2}{\pi_{22}^2+c_2} & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Or in a detailed form

$$\begin{bmatrix} \pi_{11}^1 + c_1 & \pi_{12}^1 & \pi_{13}^1 & \cdots & \pi_{1n}^1 \\ 0 & \pi_{22}^2 + c_2 & \pi_{23}^2 & \cdots & \pi_{2n}^2 \\ 0 & 0 & \pi_{33}^3 + c_3 & \cdots & \pi_{3n}^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \pi_{n3}^3 & \cdots & \pi_{nn}^3 + c_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} d_1^1 \\ d_2^2 \\ d_3^3 \\ \vdots \\ d_n^3 \end{bmatrix}$$

where

$$\pi_{ij}^{3} \equiv \pi_{ij}^{2} - \frac{\pi_{i2}^{2}\pi_{2j}^{2}}{\pi_{22}^{2}+c_{2}}, \quad i = 3,\ldots,n,\ i = i+1,\ldots,n,$$

$$d_{i}^{3} \equiv d_{i}^{2} - \frac{\pi_{i2}^{2}d_{i}^{2}}{\pi_{22}^{2}+c_{2}}, \quad i = 3,\ldots,n.$$

This sequential process of designing the i'th loop controller c_i followed by "rolling" it into the remaining loops by pre-multiplying the governing matrix equation is repeated until we design c_{n-1} and arrive at the last step. Using the pre-multiplying matrix

$$V_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \frac{-\pi_{(n)(n-1)}^{n-1}}{\pi_{(n-1)(n-1)}^{n-1}+c_{n-1}} & 1 \end{bmatrix}$$

we have

The detailed matrix equation in the last step shows that we have only one unknown left, the last controller c_n

$$\begin{bmatrix} \pi_{11}^{1}+c_1 & \pi_{12}^{1} & \pi_{13}^{1} & \cdots & \pi_{(1)(n\text{-}1)}^{1} & \pi_{(1)(n\text{-}1)}^{1} \\ 0 & \pi_{22}^{2}+c_2 & \pi_{23}^{2} & \cdots & \pi_{(2)(n\text{-}1)}^{2} & \pi_{(2)(n\text{-}1)}^{2} \\ 0 & 0 & \pi_{33}^{3}+c_3 & \cdots & \pi_{(3)(n\text{-}1)}^{3} & \pi_{(2)(n\text{-}1)}^{3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \pi_{(n\text{-}1)(n\text{-}1)}^{(n\text{-}1)}+c_{(n\text{-}1)} & \pi_{(n\text{-}1)(n)}^{(n\text{-}1)} \\ 0 & 0 & 0 & \cdots & 0 & \pi_{nn}^{n}+c_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} d_1^1 \\ d_2^2 \\ d_3^3 \\ \vdots \\ d_n^{n-1} \\ d_n^n \end{bmatrix}$$

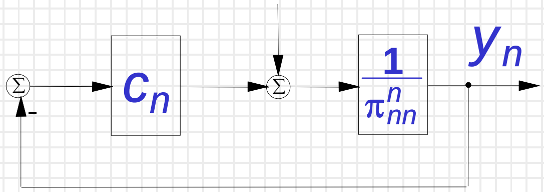
where

$$\pi_{nn}^{n} \equiv \pi_{nj}^{n-1} - \frac{\pi_{(n)(n-1)}^{(n-1)}\pi_{(n-1)(n)}^{(n-1)}}{\pi_{(n-1)(n-1)}^{(n-1)}+c_{(n-1)}},$$

$$d_n^n \equiv d_n^{(n-1)} - \frac{\pi_{(n)(n-1)}^{(n-1)}d_{(n-1)}^{(n-1)}}{\pi_{(n-1)(n-1)}^{(n-1)}+c_{(n-1)}}.$$

Clearly, the n'th output depends only on c_n

Hence, applying the performance constraint no longer requires upper bounding. The n'th step is always a simple single-loop system (shown below) that contains all plant interaction and previous n-1 designed controllers.



The design performance inequality is given by

$$\left|\frac{d_n^n}{\pi_{nn}^n + c_n}\right| \le \alpha_n(j\omega), \quad \forall P \in \mathcal{P}, \;\; \forall d \in \mathcal{d}.$$

14.5. Notes On the General Problem

1. Given our standing assumptions, stability of the closed-loop system is achieved if the last step is successfully completed. That is, c_n stabilizes the plant

$$1/\pi_{nn}^{n}.$$

2. However, successful completion of the above does not guarantee reasonable stability margins at different input-output points. For example, even if the SISO system designed at the last step has good margins, say of the form,

$$\left|\frac{1}{1+c_n\frac{1}{\pi_{nn}^{n}}}\right| \leq W_n, \quad \forall P \in \mathcal{P},$$

which guarantees a particular margin w.r.t. plant uncertainty and multivariable interactions, the design at the k'th step ($k<n$) with the margin-like spec

$$\left|\frac{1}{1+c_k\frac{1}{\pi_{kk}^{k}}}\right| \leq W_k, \quad \forall P \in \mathcal{P},$$

does not guarantee similar margin property for the *k*'th channel. This is since the plant $\left(\pi_{kk}^{k}\right)^{-1}$ does not contain information about the closed-loop dynamics of the loops yet to be closed (*k*+1 to *n*). We overcome this difficulty by using over bounding to meet performance specs, but this approach cannot be applied for the margins spec. We will see how to resolve this problem later (*Yaniv*, pg. 160).

$$S = (I + PC)^{-1} = \left[s_{ij} \right], s_{ii} = \frac{1}{1 + C_i \frac{1}{\pi_{ii}^2}}$$

Our earlier example can be used to illustrate this important point. The 2nd loop design dealt with the exact plant, however, the plant at the first step (first loop) IS NOT the actual plant. The result is lack of margins guarantee at the first step. Indeed, the peaking in $s_{11}(j\omega)$ surpasses its weight.