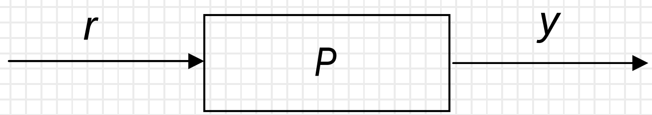
10. Multivariable Systems: Background

An m-input n-output multivariable (MIMO) system is described by



$$y = Fr$$

where

$$y(s) = \begin{bmatrix} y_1(s) \\ y_2(s) \\ \vdots \\ y_n(s) \end{bmatrix}, \quad r(s) = \begin{bmatrix} r_1(s) \\ r_2(s) \\ \vdots \\ r_m(s) \end{bmatrix}$$

$$P(s) = \begin{bmatrix} p_{11}(s) & p_{12}(s) & \cdots & p_{1m}(s) \\ p_{21}(s) & p_{22}(s) & \cdots & p_{2m}(s) \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1}(s) & p_{n2}(s) & \cdots & p_{nm}(s) \end{bmatrix}, \quad p_{ij}(s) \text{ SISO}$$

$P(s)$ is called a <u>matrix transfer function</u> (MTF).

The individual outputs are

$$y_i(s) = \begin{bmatrix} p_{i1}(s) & p_{i2}(s) & \cdots & p_{im}(s) \end{bmatrix} \begin{bmatrix} r_1(s) \\ r_2(s) \\ \vdots \\ r_m(s) \end{bmatrix} =$$

A word on notation. Some books denote vectors using lower case and matrices using upper case letters. We will use a mixed notation, as long as it does not create confusion.

10.1. Matrix Gains

The gain of a SISO system P, $y(j\omega) = P(j\omega)r(j\omega)$,

$$\frac{|y(j\omega)|}{|r(j\omega)|} = \frac{|P(j\omega)y(j\omega)|}{|r(j\omega)|} = |P(j\omega)|$$

does not depend on the input. MIMO systems are different animals.

We use vector and matrix norms to define MIMO magnitudes. For example, one measure of the magnitude of a vector x is the Euclidian norm

$$\|x(j\omega)\|_2 = \sqrt{\sum_1^n |x(j\omega)|^2} = \sqrt{x(j\omega)^H x(j\omega)}.$$

where x^H denotes conjugate transpose. The gain of the system P at a given frequency is then

which depends on the input signal.

Example: consider a 2-input 2 –output system $y = Pr$ with different inputs

$$r_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, r_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, r_3 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, r_4 = \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix}, r_5 = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}.$$

Note that all input have the same magnitude $\|r\|_2 = 1$ but have different "directions".

Now consider the plant

$$P = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}.$$

The outputs ($y = Pr$)

$$y_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, y_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, y_3 = \begin{bmatrix} 6.36 \\ 3.64 \end{bmatrix}, y_4 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}, y_5 = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}.$$

clearly depend on the input signal (direction).

Their 2-norm magnitudes are

$$\|y_1\|_2 = 5.83, \|y_2\|_2 = 4.24, \|y_3\|_2 = 7.32, \|y_4\|_2 = 1.00, \|y_5\|_2 = 0.28.$$

Unlike a SISO system, the gain of a MIMO MTF depends on the inputs.

One way to define this gain is to select the specific input vector that maximizes the gain (i.e., maximizing direction)

This gain (norm) is called the induced norm on *P* corresponding to the vector norm $\|x\|$. For example, if

the induced norm is

This is related to *Singular Values* of a matrix. Given a square P, its singular values σ_i are defined by

$$\sigma_i = \sqrt{\lambda_i(P^H P)} > 0.$$

The spectral norm is then

(the maximal singular value). We use the fact that for an *hermitian* matrix P ($x^T A x > 0$ for any $x \neq 0$, or $A^H = A$)

$$\lambda_{max} = \max_{\|x\|\neq 0} \frac{x^H P x}{\|x\|^2} = \max_{\|x\|\neq 0} \frac{x^H P x}{x^H x}$$

$$\lambda_{min} = \min_{\|x\|\neq 0} \frac{x^H P x}{\|x\|^2} = \min_{\|x\|\neq 0} \frac{x^H P x}{x^H x}.$$

Also, the minimum gain is

$$\min_{\|x\|\neq 0} \frac{\|Px\|_2}{\|x\|_2} = \min_{\|x\|\neq 0} \sqrt{\frac{x^H P^H P x}{x^H x}} =$$

$\|x\|\neq 0$

(the minimal singular value).

The singular values of $P(j\omega)$ are known as *principal gains*. It can be shown that the gain of a matrix $P(j\omega)$ is bounded by

The spectral radius of P is

$$\rho(P) = \max_i |\lambda_i(P)| \leq \|P\|.$$

10.2. Systems

The 2-norm of a function $y(s)$ is defined by

$$\|y\|_2 = \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} y(j\omega)^H y(j\omega)\mathrm{d}\omega\right]^{0.5}.$$

It can be shown that for a linear system $y(s) = P(s)r(s)$

where (the H_∞ norm)

In a SISO system

$$\|P\|_\infty = \max_\omega |P(j\omega)|.$$

10.3. Poles and Zeros

If p is a pole of a SISO $P(s)$

$$P(s) = \frac{n(s)}{d(s)}$$

then

Hence, the poles of $P(s)$ are the roots of the denominator $d(s)$. In a MIMO system

$$P(s) = \frac{1}{d(s)} \begin{bmatrix} n_{11}(s) & n_{12}(s) & \cdots & n_{1m}(s) \\ n_{21}(s) & n_{22}(s) & \cdots & n_{2m}(s) \\ \vdots & \vdots & \vdots & \vdots \\ n_{n1}(s) & n_{n2}(s) & \cdots & n_{nm}(s) \end{bmatrix}$$

The poles of $P(s)$ include the roots of the denominator $d(s)$. However, the roots of $d(s)$ do not reveal multiplicity of poles of $P(s)$.

If z is a zero of a SISO $P(s)$

$$P(s) = \frac{n(s)}{d(s)}$$

then

Hence, the zeros of $P(s)$ are the roots of the numerator $n(s)$. In a MIMO system, z is a zero of $P(s)$ if the rank of $P(s)$ is less than its normal rank. This means that there exists at least one constant vector $v \notin 0$ such that

$$P(z)\mathrm{v} = 0.$$

and at least one constant vector $w \notin 0$ such that

$$w^T P(z) = 0.$$

v and w^T are part of nullspaces generated by rows and columns of $P(z)$.

Note that the roots of $\det P(s) = 0$ may not be all the zeros. This is due to pole/zero cancellations when the minors are formed.

Now consider a system input

$$r(s) = v\,\frac{1}{s-z}$$

The output is given by

This input is blocked from the output! We call these *transmission zeros*.

Example: using MATLAB to computes MIMO poles/zeros. The system

$$P(s) = \frac{1}{s+1}\begin{bmatrix} s+3 & 2 \\ 3 & 1 \end{bmatrix}$$

has poles at [-1,-1] and a zero a [3] (even though $n_{ij}(s)$ do not). This system is said to be non-minimum phase. To compute these using MATLAB:

```
>>d = [1,1];
>>G = tf({[1,3],2;3,1},{d,d;d,d})

Transfer function from input 1 to output...
      s + 3
 #1:  -----
      s + 1

        3
 #2:  -----
      s + 1

Transfer function from input 2 to output...
        2
 #1:  -----
      s + 1

        1
 #2:  -----
      s + 1
```

The poles are the eigenvalues of the state-space realization

```
>> eig(G)
ans =
    -1
    -1
```

And the zeros are computed from

```
>> tzero(G)
ans =
     3
```

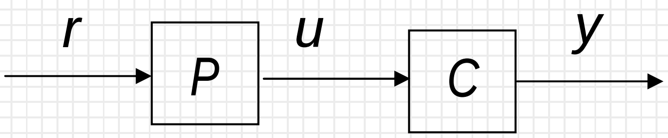
Note that

$$P(s=3)=\frac{1}{3+1}\begin{bmatrix}3+3 & 2\\ 3 & 1\end{bmatrix}=\frac{1}{3+1}\begin{bmatrix}6 & 2\\ 3 & 1\end{bmatrix}$$

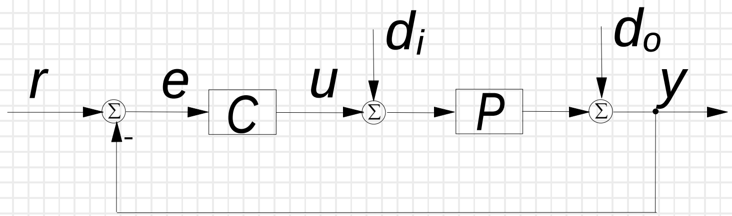
with the rank of *P* dropping from 2 to 1.

10.4. Basic Operations

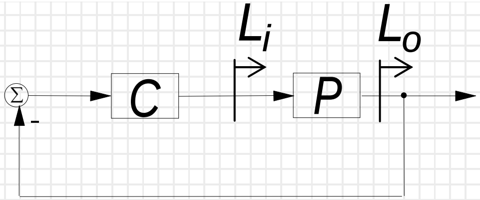
In general, MTFs do not commute. For example, in the cascade system



and in a feedback system



The open-loop MTF is defined based on the location where the loop is opened.



If the loop is opened at the plant output, then

appears in the sensitivity and complimentary sensitivity functions

which are sometimes referred to as <u>output</u> sensitivity and <u>output</u> complimentary sensitivity.

If we break the loop at the plant input we obtain

with the corresponding <u>input</u> sensitivity and <u>input</u> complimentary sensitivity functions

For example,

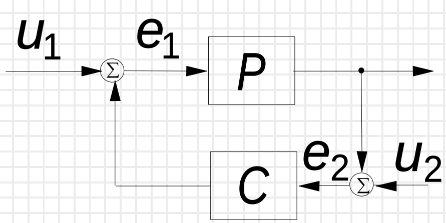
Some useful relations:

To show the above, we use the push-through rule (assuming appropriate matrix dimensions)

10.5. Internal Stability

Definition. A rational MTF is exponentially stable iff it is proper and has no poles in the closed RHP.

Definition. The feedback system shown below is *internally stable* iff the MTF from u to e



is exponentially stable.

The above is needed to exclude unstable pole/zero cancellations which cannot be detected by Nyquist-like stability results. Hence, we need to show that each is exponentially stable.

If both P and C are stable, it is sufficient to check just one.

If only one is stable, we need to check only a specific MTF for stability. This case is present next.

Theorem. If C is exponentially stable, then the feedback system shown in previous page is internally stable iff

is exponentially stable.

Proof. (only if): immediate.

(if):

So if C and H_{21} are exponentially stable, then so is H_{11}. It follows that

is also exponentially stable.

Finally,

Which shows that H_{22} is also exp. stable. Hence, the feedback system is internally stable.

Example. Let

$$P = \frac{1}{s-1}, \quad C = \frac{s-1}{s+2}$$

Having one closed-loop transfer function stable

does not reveal the whole story since

is unstable.

10.6. Nyquist-Like Stability Results

Theorem. *If C is exp. stable, then H_{21} is exponentially stable iff*

1) det(I+PC) has no zeros in closed RHP (including infinity), and

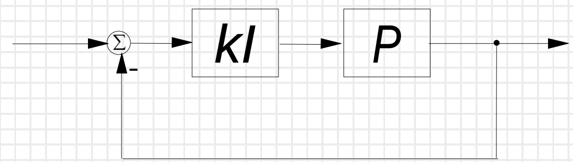
2) (I+PC)P has no unstable pole-zero cancellations.

Proof. See Maciejowski (pg. 57).

The added condition 2) is needed in automatic synthesis techniques such as LQG and H_∞. When using manual loop-shaping, simply avoid introducing unstable zeros and poles in the controller that coincide with those in the plant.

Note that in a SISO system det(I+PC) = +PC is the characteristic equation.

Next, we proceed with a generalization of Nyquist stability criterion. We consider a square, rational MTF P in a series with a gain compensator $K = kI$ as shown to the right.



Recall the *Principle of the Argument* from Ch 6. Let $\Gamma(s)$ be the Nyquist counter with $j\omega$-axis indentations for poles of $\det(I+kP(s))$. Let $\det(I+kP(s))$ have n_p poles and n_z zeros inside $\Gamma(s)$. Then as s traces $\partial\Gamma$ once counterclockwise

$$N = n_z - n_p$$

where N denotes the no. of counterclockwise origin encirclements by the plot of $\det(I+kP(s))$.

In a SISO setting, it is sufficient to obtain the plot for a single k, then infer stability properties for all $k \geq 0$.

The difficulty in a MIMO setting, is that since we're dealing with matrices, we would have to plot det($I+kP(s)$) for each k of interest.

The *characteristic loci* is now used to overcome this difficulty. They involve eigenvalues as follows.

Let $\lambda_i(s)$ be an eigenvalue of $P(s)$, so by definition

where u denotes an eigenvector. Also

implies that $k\lambda_i(s)$ be an eigenvalue of $kP(s)$. So,

shows that $1+k\lambda_i(s)$ is an eigenvalue of $I+kP(s)$.

Finally, since the determinant equals the product of the eigenvalues

That is, we can study closed-loop stability properties by counting total number of origin encirclements made by the Nyquist plots of $1+k\lambda_i(s)$. Equivalently, we can count the total number of -1 encirclements made by the Nyquist plots of $k\lambda_i(s)$.

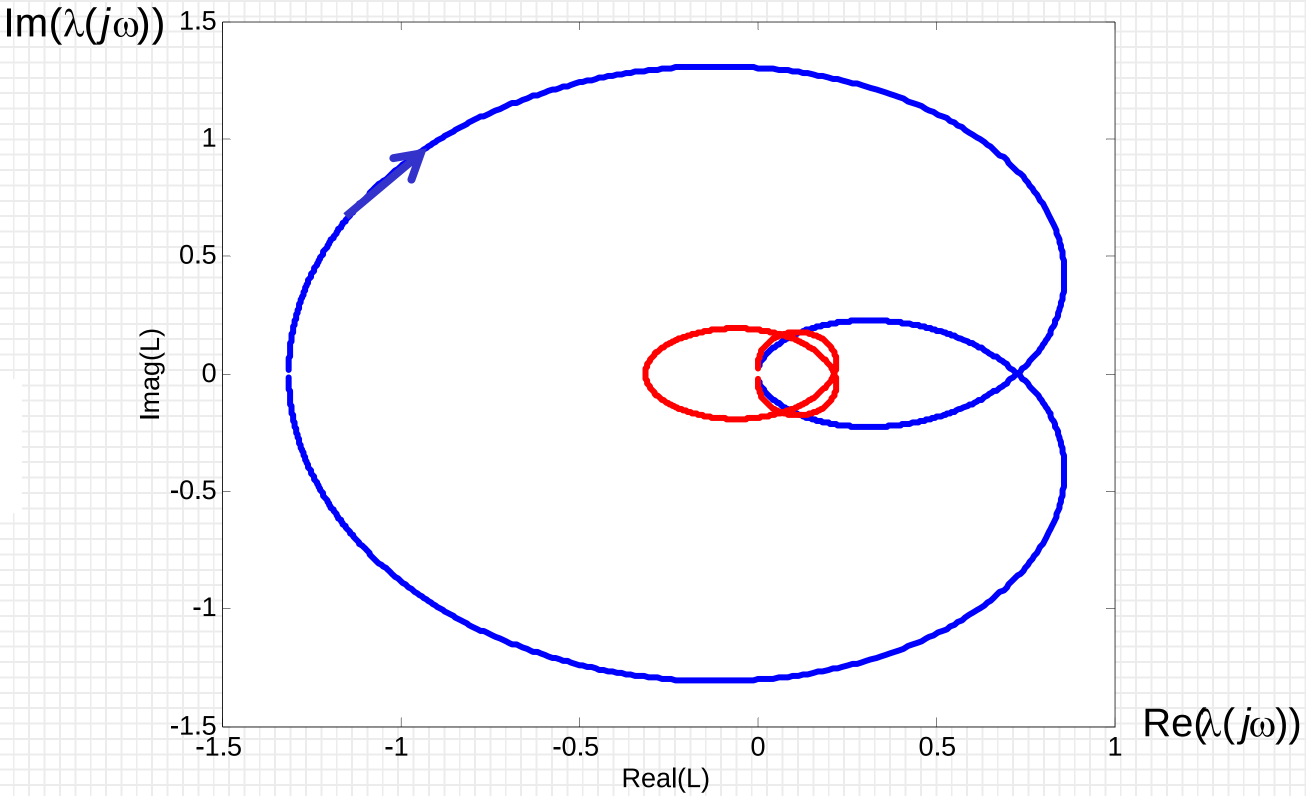
These plots of $\lambda_i(s)$ are called characteristic loci.

Note that each Ny. plot of $k\lambda_i(s)$ may not be a closed curve (eigenvalues are not rational functions). But when the characteristic loci are drawn together they form a closed curve.

Generalized Nyquist Stability Criterion. *Let L(s) have* n_p *unstable (MIMO) poles. The close-loop system with unity negative feedback and open-loop kL(s) (k>0) is stable iff the eigenloci of kL(s) encircles the point (-1,0)* n_p *time CCW (assuming no hidden unstable modes).*

Example: plot the characteristic loci (`cgloci`) for

$$P = \frac{1}{(s+2)(s+4)}\begin{bmatrix} s-1 & 2 \\ s & s-3 \end{bmatrix}, \quad C = \begin{bmatrix} 2.5 & 0 \\ 0 & 3 \end{bmatrix}$$



10.7. Diagonal Dominance

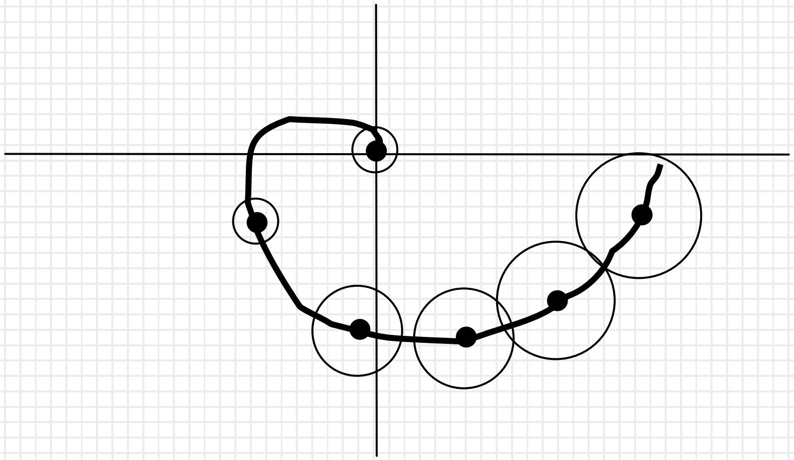
Gershgorin's Theorem. Let $Z = [z_{ij}]$ be a, mxm complex matrix. Then its eigenvalues lie in the union of m circles, each with center z_{ii} and radius

$$\sum_{\substack{j=1 \\ j\neq i}}^{m} |z_{ij}|, \quad i = 1,\ldots m, \quad \text{or} \quad \sum_{\substack{j=1 \\ j\neq i}}^{m} |z_{ji}|, \quad i = 1,\ldots m.$$

Now let $Z = P(s)$. At the loci $p_{ii}(j\omega)$ superimpose (pointwise) a circle of radius

$$\sum_{\substack{j=1 \\ j\neq i}}^{m} |p_{ij}(j\omega)| \quad \text{or} \quad \sum_{\substack{j=1 \\ j\neq i}}^{m} |p_{ji}(j\omega)|$$

The union of the Gershgorin bands include the characteristic loci as illustrated below.



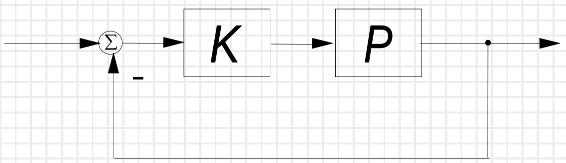
Hence, we can assess closed-loop stability by counting encirclements of the (-1,0) point by the Geshsgorin bands.

Definition. The MTF is said to be *diagonally dominant* if

$$|p_{ii}(j\omega)| > \sum_{\substack{j=1 \\ j\neq i}}^{m} |p_{ij}(j\omega)|, \text{ or } \quad |p_{ii}(j\omega)| > \sum_{\substack{j=1 \\ j\neq i}}^{m} |p_{ji}(j\omega)|, \quad i = 1,\dots,m,\ \omega \geq 0.$$

The following result exploit diagonal dominance.

Theorem (Rosenbrock, 1970). Suppose that P is square, that $K = \text{diag}\{k_i,\ldots,k_m\}$, and that



and let the i'th Gershgorin band of $P(j\omega)$ encircle the point $-1/k_i$ N_i CCW. Then the closed-loop system shown above is stable iff

$$\sum_i N_i = n_p$$

where n_p is the number of unstable poles of $P(s)$ (assuming no unstable hidden modes).

The above can also be stated in terms of column dominance (even pointwise).

Note that the gains k_i in each loop may be different.

Such Nyquist-array-based stability tests are sufficient, but not necessary. If the bands overlap the -1 point (i.e., we do not have diagonal dominance), we cannot decide whether the system is stable or not.

It is therefore useful to have diagonal dominance. Note that eigenvalues are invariant under similarity transformation

$$\lambda(Y) = \lambda(XYX^{-1}).$$

The idea is to find a scaling matrix X that such that XYX^{-1} is diagonally dominant. This can be done at a fixed frequency, but is more difficult to achieve at all frequencies.

Note that

$$X(\mathbf{I}+\mathbf{L})\mathbf{X}^{-1} = I + X\mathbf{L}\mathbf{X}^{-1}$$

allows us to infer stability from the bands of XYX^{-1}.

10.8. References

- Goodwin, Graebe and Salgado, *Control System Design*, 2001.
- Morari and Zafiriou, *Robust Process Control*, 1989.
- Maciejowski, *Multivariable Feedback Design*, 1989.
- Skogestad and Postlethwaite, *Multivariable Feedback Control*, 1996.

- Sidi, *Design of Robust Control Systems from classical to modern practical approaches*, 2001.
- Rosenbrock, *State-Space and Multivariable Theory*, 1970.

10.9. Homework

1. Prove the relations on page 16.
2. Show that for the system on page 17

$$\begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix} = \begin{bmatrix} I & -C(s) \\ -P(s) & I \end{bmatrix}^{-1}$$