12. Discrete-Time Control[1,2]

There are several methods for designing digital controllers:

- Continuous-time design followed by controller discretization
- Frequency response in w domain
- Direct design in z domain (e.g., root locus and QFT Toolbox)

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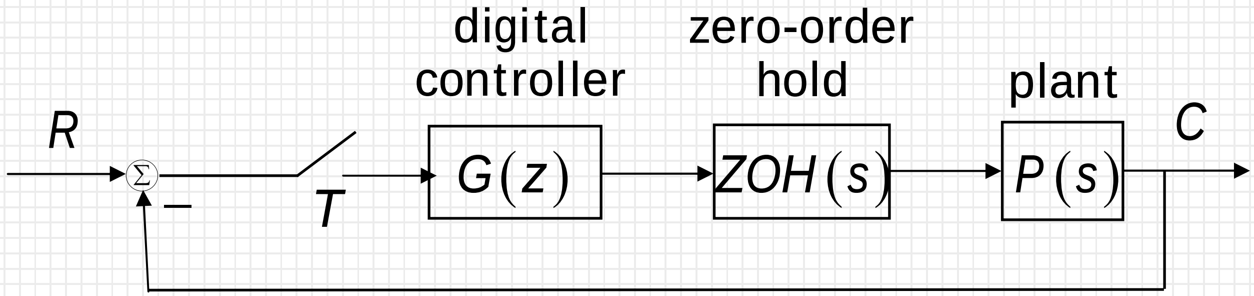
1. Ogata, K., *Discrete-Time Control Systems*, Prentice-Hall, Inc., 1987.
2. Houpis, CH., and Rasmussen, SJ., *Quantitative Feedback Theory Fundamentals and Applications*, Marcel Dekker AG, 1999.

12.1. Controller Discretization

We assume the system is LTI and use any LTI design technique, then discretize the controller via several methods.

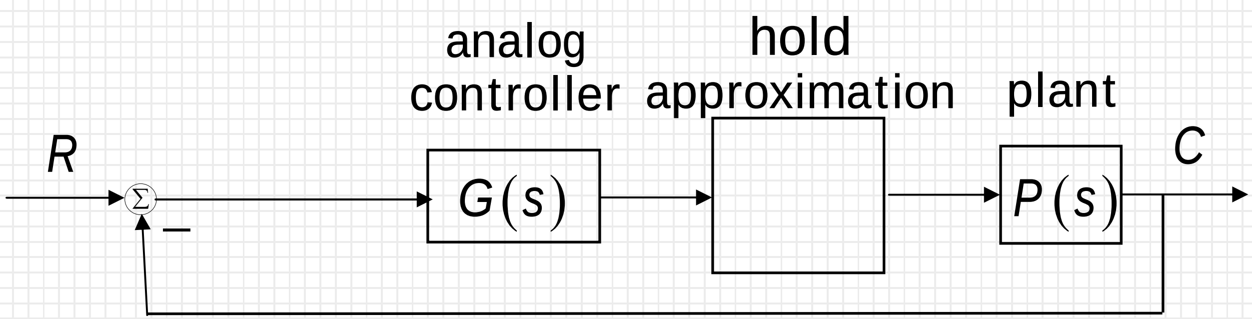
The hold circuit converts digital controller output from the digital controller into an analog signal. The most common hold is the ZOH circuit. A digital control system (top) and its approximation in a continuous-time setting (bottom) are shown below.



The hold circuit introduces delay and attenuates at high frequencies. Pade` approximation often replaces the delay during design:

This approx. is good for $\omega < \omega_s/10$. A 2nd-order approx. is good for $\omega < \omega_s/3$. Taking into account the sampler, the ZOH is replaced with

The equivalent continuous-time system is shown below.



The analog controller can now be design using any of many techniques. We then discretize it and verify performance of the hybrid control system. Discretization methods include

- Numerical algorithms
- Transient response invariance
- Matched pole-zero locations

12.1.1. Numerical Algorithms

Consider a simple 1s order filter (controller)

$$\frac{U}{E}(s) = \frac{a}{s+a}$$

or in its differential equation form

$$\dot{u}(t) + au(t) = ae(t).$$

Integrating both sides

$$\int_0^t \dot{u}(t)dt + \int_0^t au(t)dt = \int_0^t ae(t)dt.$$

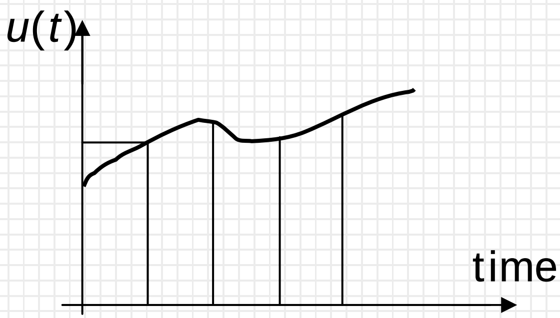
Let $t = kT$ with T the sampling period

$$u(kT) - u(0) = -a\int_0^{kT} u(t)\,dt + a\int_0^{kT} e(t)\,dt$$

with a similar relation at $t = (k\text{-}1)T$. So

The above integrals are now evaluated using numerical integration – backward difference. Specifically,

Applying this to our equation above gives



and taking z transform results in

$$U(z) = z^{-1}U(z) - aT\left(U(z) - E(z)\right)$$

leading to this transfer function

which, when compared with the original $U/E=1/(s+a)$ implies

The same result is obtained using

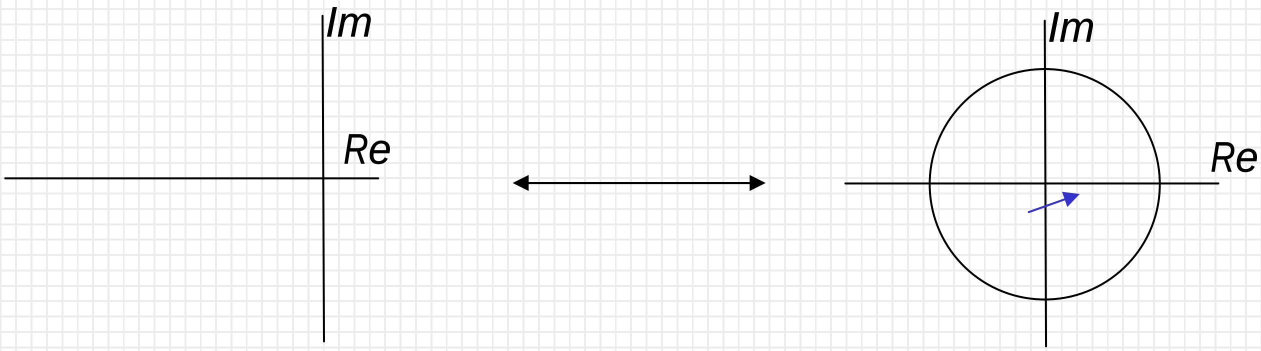
$$\frac{du}{dt} = \frac{u(kT) - u\left((k-1)T\right)}{T}$$

$$\Rightarrow u(kT) = u\left((k-1)T\right) - aT\left(u(kT) - e(kT)\right).$$

Is the mapped filter stable? In the s domain we require that all poles lie in Re[s] < 0, that is

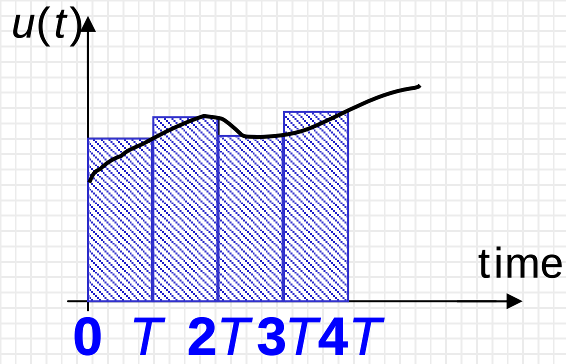
Let $z = \sigma + j\omega$, since $T > 0$

$$\mathrm{Re}\left[\frac{\sigma + j\omega - 1}{\sigma + j\omega}\right] < 0 \Rightarrow \frac{\sigma^2 - \sigma + \omega^2}{\sigma^2 + \omega^2}$$



We conclude that this mapping takes stable LTI filters into a stable z domain filters. However, there are frequency domain distortions.

Another numerical integration method – trapeziodal – leads to a better map



Hence

$$u(kT) = u((k-1)T) - \tfrac{1}{2}aT\left(u(kT) - u((k-1)T)\right) + \tfrac{1}{2}aT\left(e(kT) - e((k-1)T)\right)$$

whose z transform is

$$U(z) = z^{-1}U(z) - \tfrac{1}{2}aT\left(U(z) - z^{-1}U(z)\right) + \tfrac{1}{2}aT\left(E(z) - z^{-1}E(z)\right)$$

leading to the transfer function

and it follows that the s to z mapping is

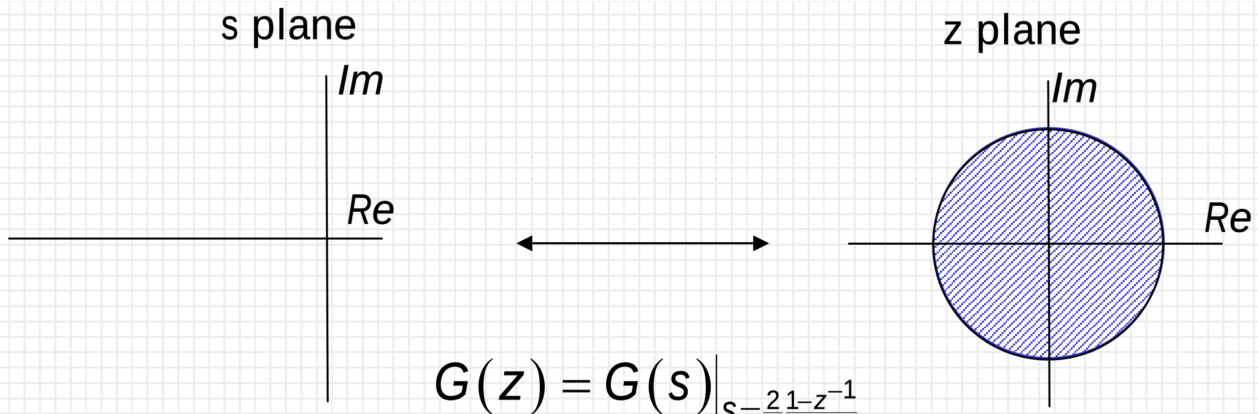
Note that

$$G(z) = G(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

has the same number of poles and zeros, and the same number of poles as those of $G(s)$.

Let's study stability of the mapping:

which can be reduced to



$$G(z) = G(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}}$$

Hence, the trapeziodal integration (better known as *bilinear* or *Tustin* transformation) preserves stability.

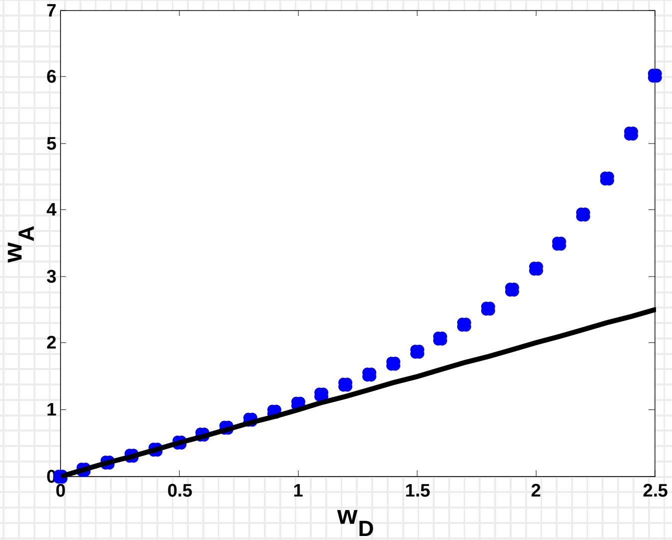
Let's compare the frequency responses of $G(s)$ and $G(z)$, that is, of $G(j\omega)$ and $G\left(z = e^{j\omega T}\right)$. Let $s = j\omega_A$ and $z = e^{j\omega_D T}$

That is

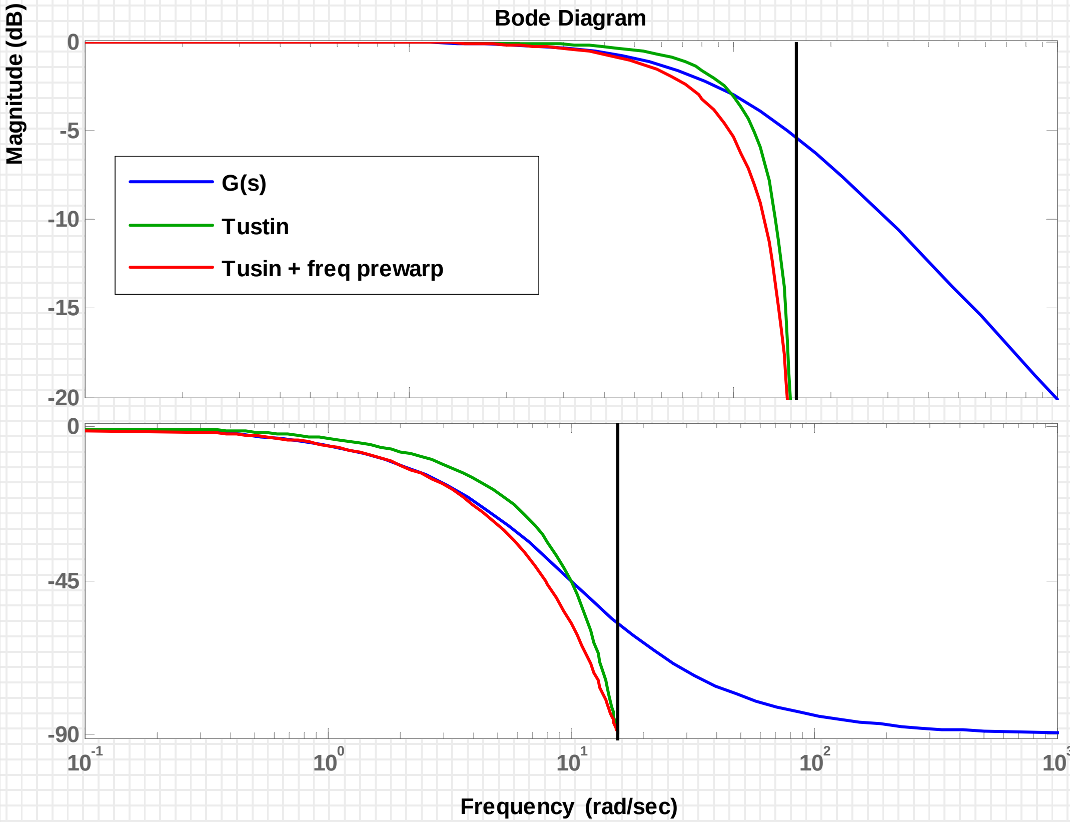
When $\omega_D T$ is small then

implying similar frequency responses at "low" frequencies. However, $\omega_A \xrightarrow[\omega_D \to \frac{1}{2}\omega_s]{} \infty$ indicating frequency distortions.

To minimize such distortion, we can adjust the frequency scale by shifting the corner frequency as follows



$$G(s)=\frac{10}{s+10}\Rightarrow G(z)\underset{T=0.2}{=}\begin{cases}\dfrac{0.5(z+1)}{z} & \text{tustin}\\[2ex] \dfrac{0.609(z+1)}{(z+0.218)} & \text{tustin with freq prewarp}\end{cases}$$




```
P=tf(10,[1,10]);

P1=c2d(P,.2,'tustin');

P2=c2d(P,.2,'prewarp',10);


P1=tf([.5,1],[1,0],'Ts',.2);
>> get(P1)
       num: {[0.609 0.609]}
       den: {[1 0.218]}
  Variable: 'z'
        Ts: 0.2
   ioDelay: 0
InputDelay: 0
OutputDelay: 0
 InputName: {''}
OutputName: {''}
InputGroup: [1x1 struct]
OutputGroup: [1x1 struct]
     Notes: {}
  UserData: []
```

12.1.2. Transient Response Invariance

We cover here only *step invariance* method. The goal is to have

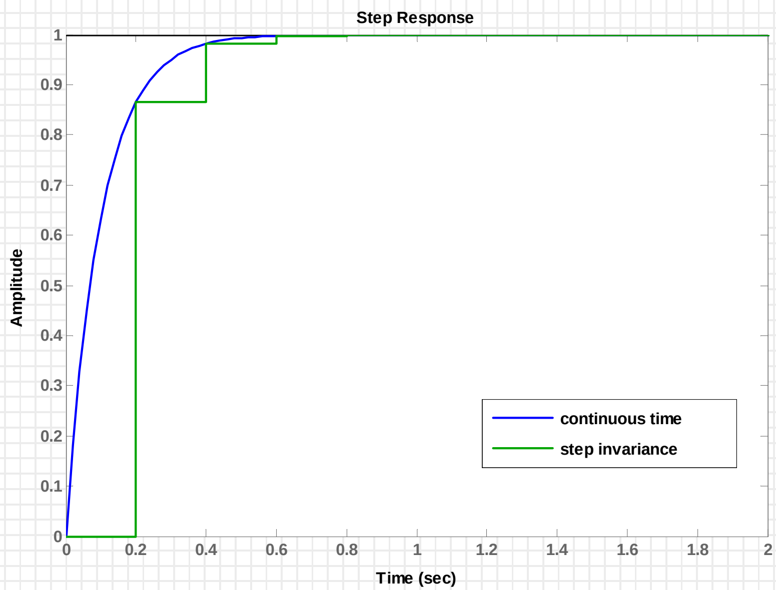
$$\mathcal{Z}^{-1}\left[G(z)\frac{1}{1-z^{-1}}\right]=\mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]_{t=kT}.$$

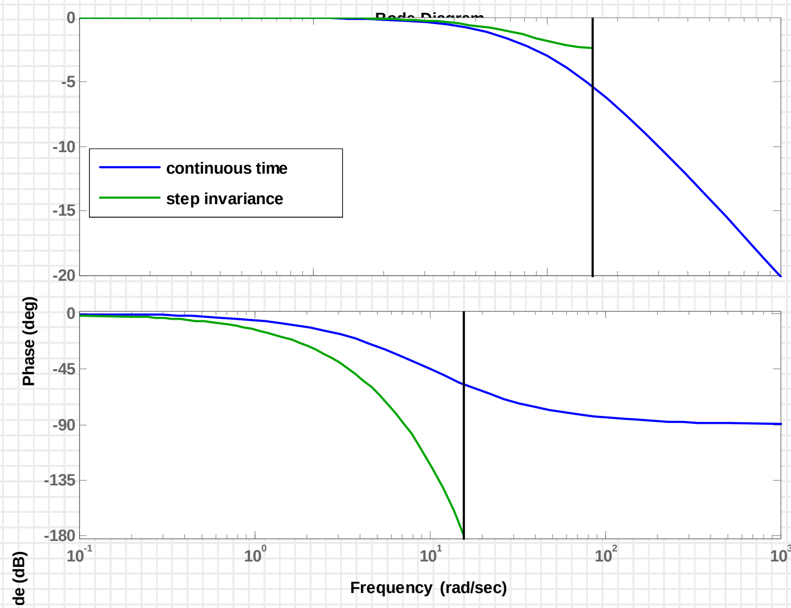
Taking z transform of both sides

or

Hence, the step response of $G(s)$ at $t = kT$ is equal to the step response of $G(z)$. In our 1st order filter

$$G(z)=\left(1-z^{-1}\right)\mathcal{Z}\left[\frac{a}{s(s+a)}\right]=\frac{\left(1-e^{-aT}\right)z^{-1}}{1-e^{-aT}z^{-1}}$$





Aliasing of $G(j\omega)$ at frequency beyond Nyquist frequency can be a problem, but the $1/s$ term helps attenuate these effects. Also, stability is preserved.

Other discretization methods such as impulse response invariance and matched pole-zero are used.

```
>>SYSD = C2D(SYSC,Ts,METHOD) converts the
continuous-time LTI model SYSC to a
discrete-time model SYSD with sample time
Ts.
    The string METHOD selects the
discretization method among the following:
      'zoh' Zero-order hold on the inputs
      'foh' Linear interpolation of inputs
            (triangle appx.)
      'imp' Impulse-invariant discretization
      'tustin' Bilinear (Tustin)
               approximation
      'prewarp' Tustin approximation with
                frequency prewarping. The
                critical frequency Wc (in
                rad/sec) is specified as
                fourth input by
          SYSD = C2D(SYSC,Ts,'prewarp',Wc)
      'matched' Matched pole-zero method
                (for SISO systems only).
```


To summarize, the equivalent continuous-time design involves an approximation of the hold circuit, design of an LTI controller and its discretization, followed by simulations. This scheme works with uncertain plants.

If sampling frequency is not sufficiently fast with respect to plant dynamics we must use direct z domain techniques as presented next.

12.2. w-Plane Design

As discussed earlier, the s to z mapping distorts the dynamics response in the digital domain. For example, the simple asymptotic behavior of 1st order pole, lead/lad elements and others is not maintained with the map $z = e^{j\omega T}$. The $j\omega$-axis is mapped into the unit circle $|z| = 1$.

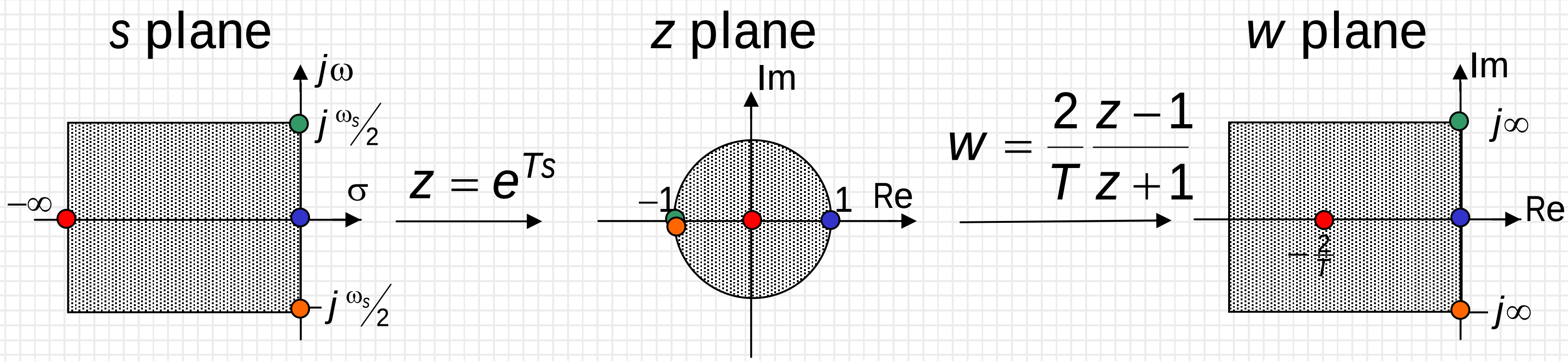
As we saw in Chap. 11, the map

$$w = \frac{z-1}{z+1}$$

takes the primary strip into an entire left-half complex plane. This potentially imply that s plane frequency response dynamics carry over to the w plane. However,

We can fix this problem using

Graphically, we have

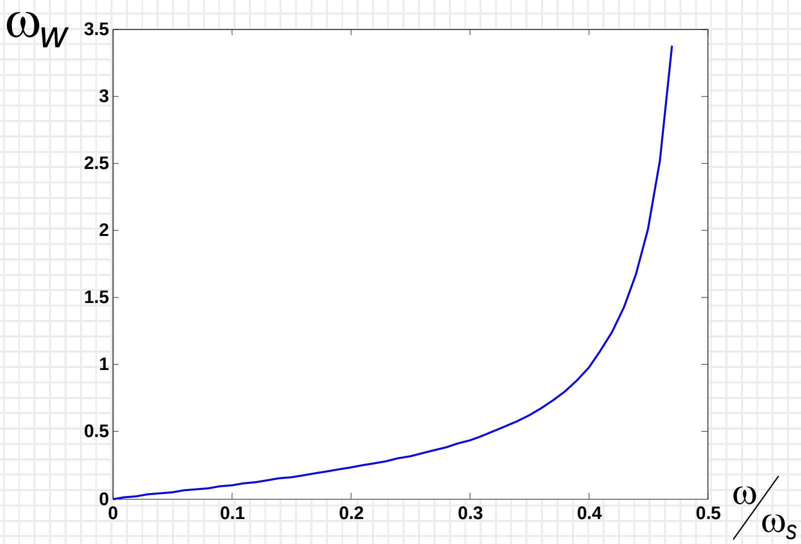


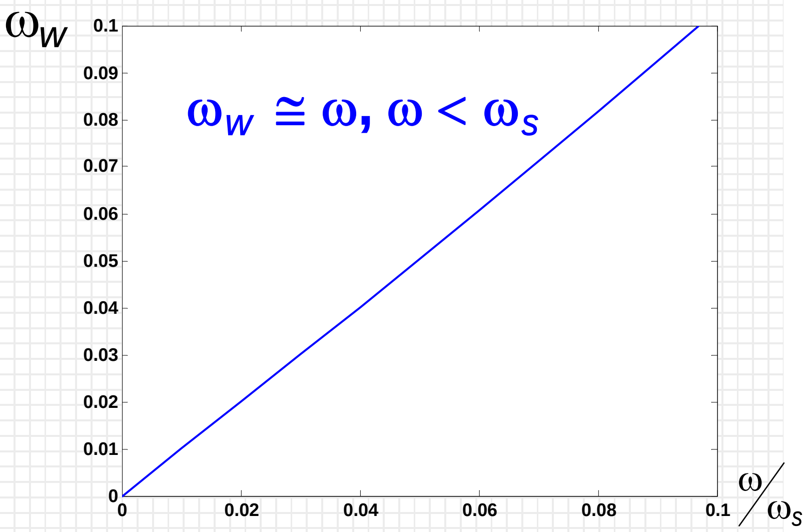
Next we compare the complex number $w = \sigma_w + j\omega_w$ with $s = \sigma + j\omega$

A similar relation can be developed for the real axis

$$\sigma_w = \frac{2}{T}\tanh\frac{\sigma T}{2}.$$

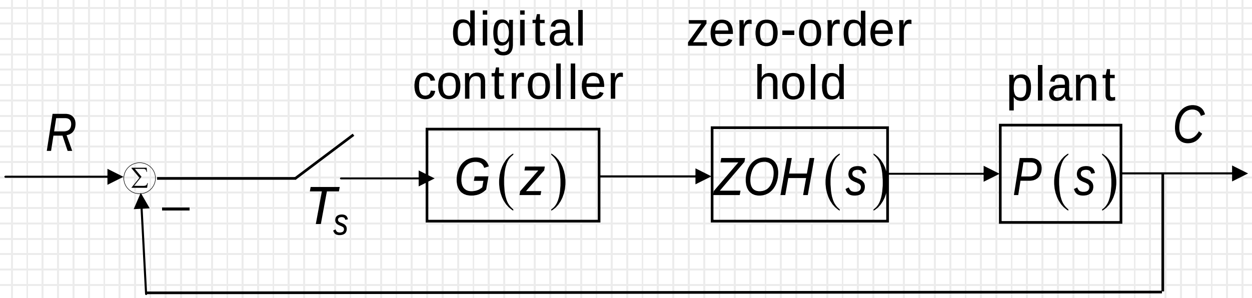
As ω increases from 0 to $\omega_s/2$, ω_w increases from 0 to ∞.





To summarize,

Let us proceed with a constructive design example.



where

$$\mathcal{P} = \left\{ P(s) = \frac{k}{s(s+a)} : \quad k \in [1,10], \;\; a \in [1,10] \right\}$$

and the specs

$$\left| \frac{C^*}{R^*} \right| = \left| \frac{(P_{ZOH})^* G^*}{1+(P_{ZOH})^* G^*} \right| \leq 2, \; \forall P \in \mathcal{P}$$

$$\left| L^*(j1) \right| \geq 10.$$

We first transform the problem into the z domain

$$P_{ZOH}(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left[\frac{k}{s^2(s+a)} \right] = \left(1 - z^{-1}\right) \mathcal{Z}\left[\frac{k/a}{s^2} + \frac{-k/a^2}{s} + \frac{k/a^2}{(s+a)} \right]$$

$$= \frac{z-1}{z}\left(\frac{k}{a}\frac{T_s z}{(z-1)^2} - \frac{k}{a^2}\frac{z}{(z-1)} + \frac{k}{a^2}\frac{z}{z-e^{-aT_s}}\right) = \frac{k}{a}\left(\frac{T_s}{z-1} - \frac{1}{a} + \frac{1}{a}\frac{z-1}{z-e^{-aT_s}}\right).$$

Next, transform into the w domain using

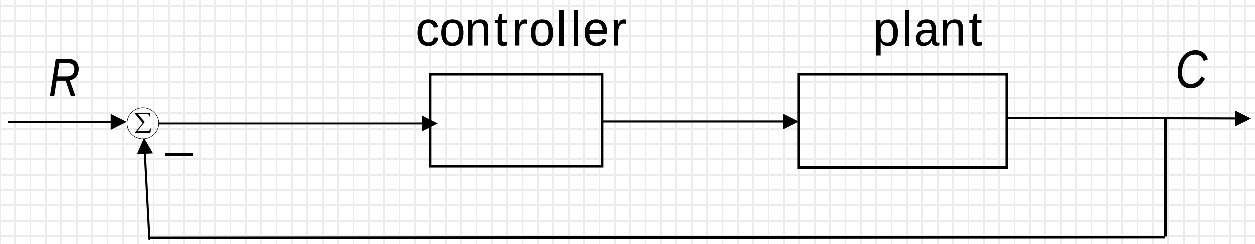
$$= \frac{k}{a}\left(\frac{T_s}{\frac{T_s w+2}{-T_s w+2}-1} - \frac{1}{a} + \frac{1}{a}\frac{\frac{T_s w+2}{-T_s w+2}-1}{\frac{T_s w+2}{-T_s w+2}-e^{-aT_s}}\right)$$

$$= \frac{w^2\left(\frac{Ts}{2} - \frac{1}{a} + \frac{1}{a}\frac{2}{1+e^{-aT_s}}\right) + w\left(-\frac{1-e^{-aT_s}}{1+e^{-aT_s}} + 1 - \frac{1}{a}\frac{1-e^{-aT_s}}{1+e^{-aT_s}}\right) + \frac{2}{T_s}\frac{1-e^{-aT_s}}{1+e^{-aT_s}}}{w\left(w + \frac{2}{T_s}\frac{1-e^{-aT_s}}{1+e^{-aT_s}}\right)}$$

As noted earlier, the above is a proper transfer function. Also

$$\left.\begin{array}{l} \left(\frac{\sigma T_s}{2}\right)^2 < 2 \\ \frac{\omega T_s}{2} < 0.297 \end{array}\right\} \Rightarrow s \approx w$$

The problem is been transformed into the w domain.



with specs (ignoring frequency distortion)

$$\left|\frac{C}{R}(w)\right| = \left|\frac{P_{ZOH}(w)G(w)}{1+P_{ZOH}(w)G(w)}\right| \leq 2, \forall P \in \mathcal{P}$$

$$|L(w)| \geq 10, \; w = j\omega_w = j1.$$

Implicit is the assumption

$$\left(\frac{\sigma T_s}{2}\right)^2 < 2$$

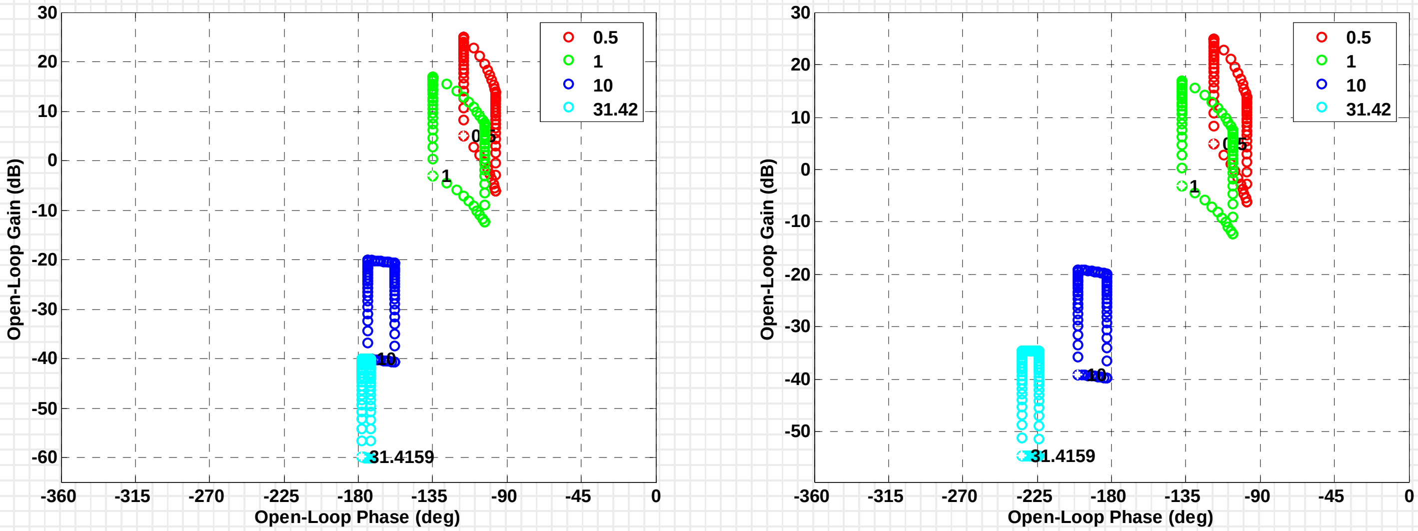
$$\frac{\omega T_s}{2} < 0.297.$$

What sampling time should be selected? If it's up to you, then you want the smallest possible time.

The second inequality implies

$$T_s < \frac{2 \times 0.297}{\omega} \simeq 0.6.$$

But this upper bound seem too slow. To see this, let and compare templates. We consider frequencies up to



Next, we compute bounds, then loop shaping. Let's check one things before we loop shape. The nominal plant is (k = a =1) which in most cases (strictly proper $P(s)$) has an NMP zero at

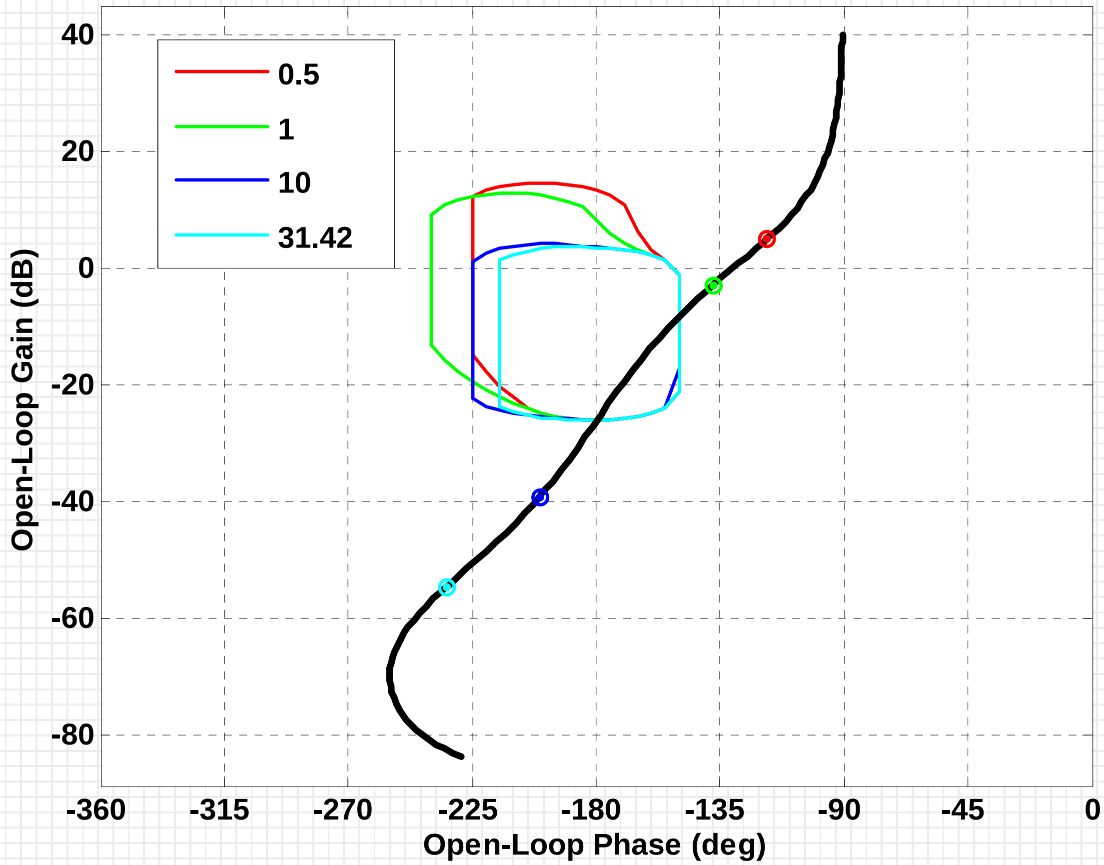
originating from the $(1-z^{-1})$ term which has a zero at $z = 1$. Recalling our loop shaping feasibility analysis, we factor the plant as

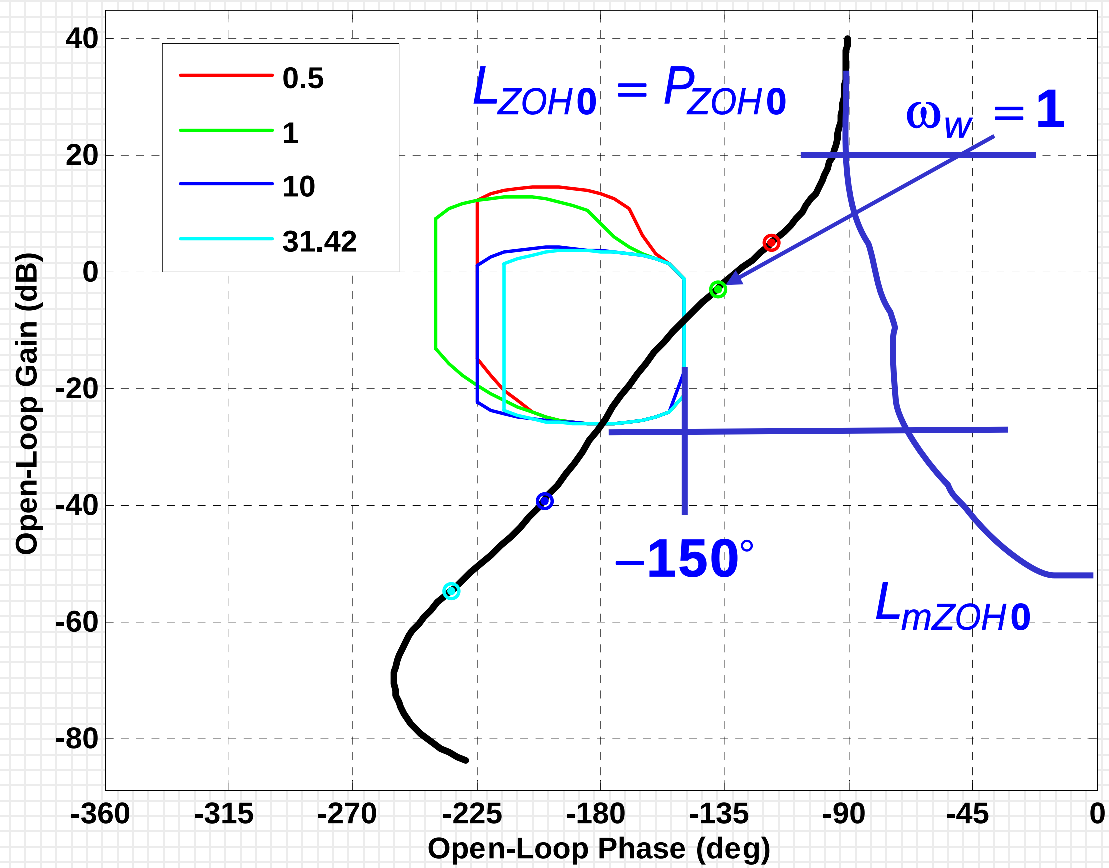
and then estimate how much gain reduction L_{mZOH0} can achieve from the spec at $\omega_w = 1$ before the all-pass phase delay "beats" it. We use the relation

$$\text{slope of } L(j\omega) \approx \frac{\angle L(j\omega)}{-90^\circ} \times \left(-20\,\frac{\text{dB}}{\text{dec}}\right).$$

Note that for w domain loop shaping we can use LPSHAPE in a continuous-time mode.

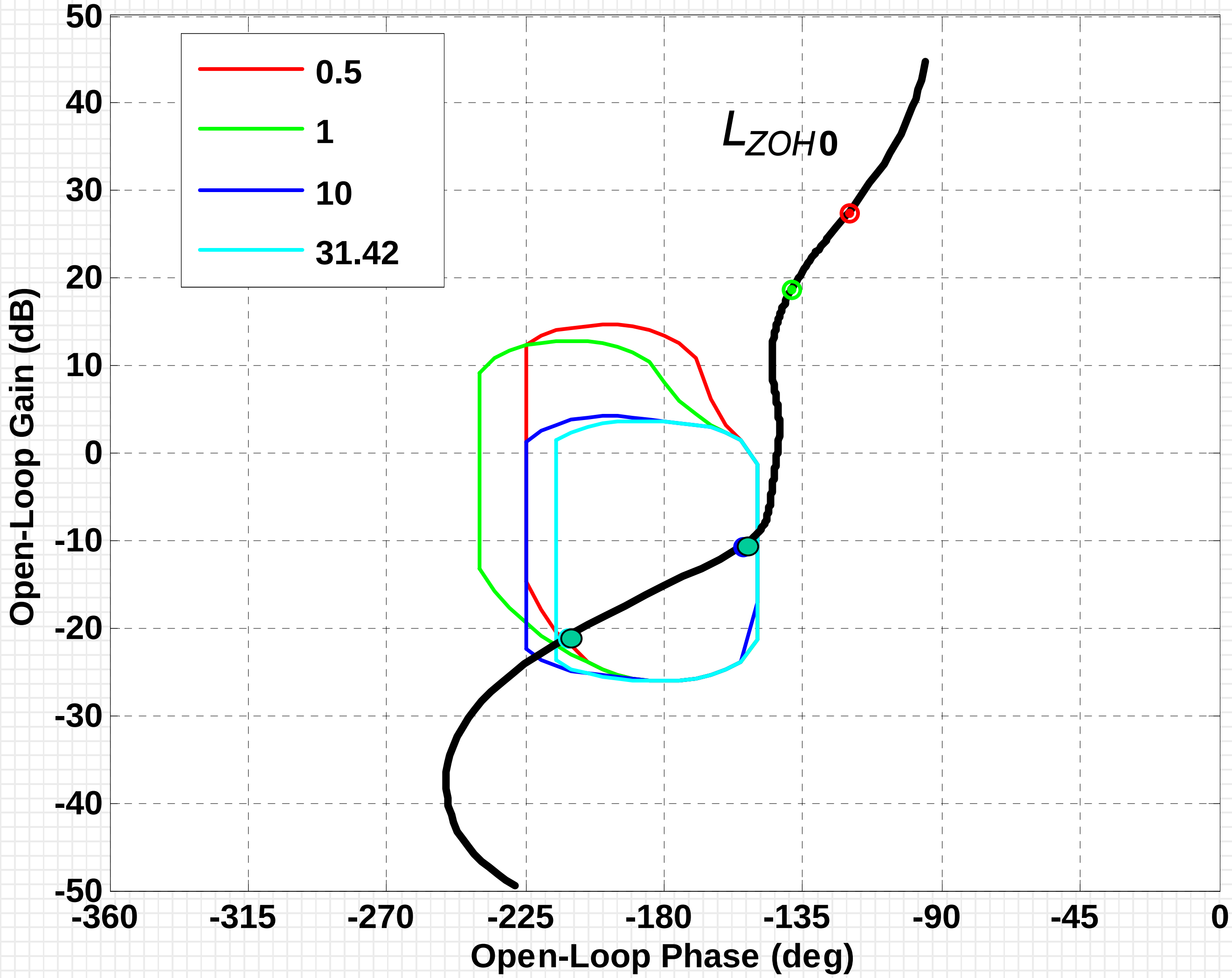
The question is can L_{ZOH0} drop from 20dB at ω_W = 1 to -25dB?





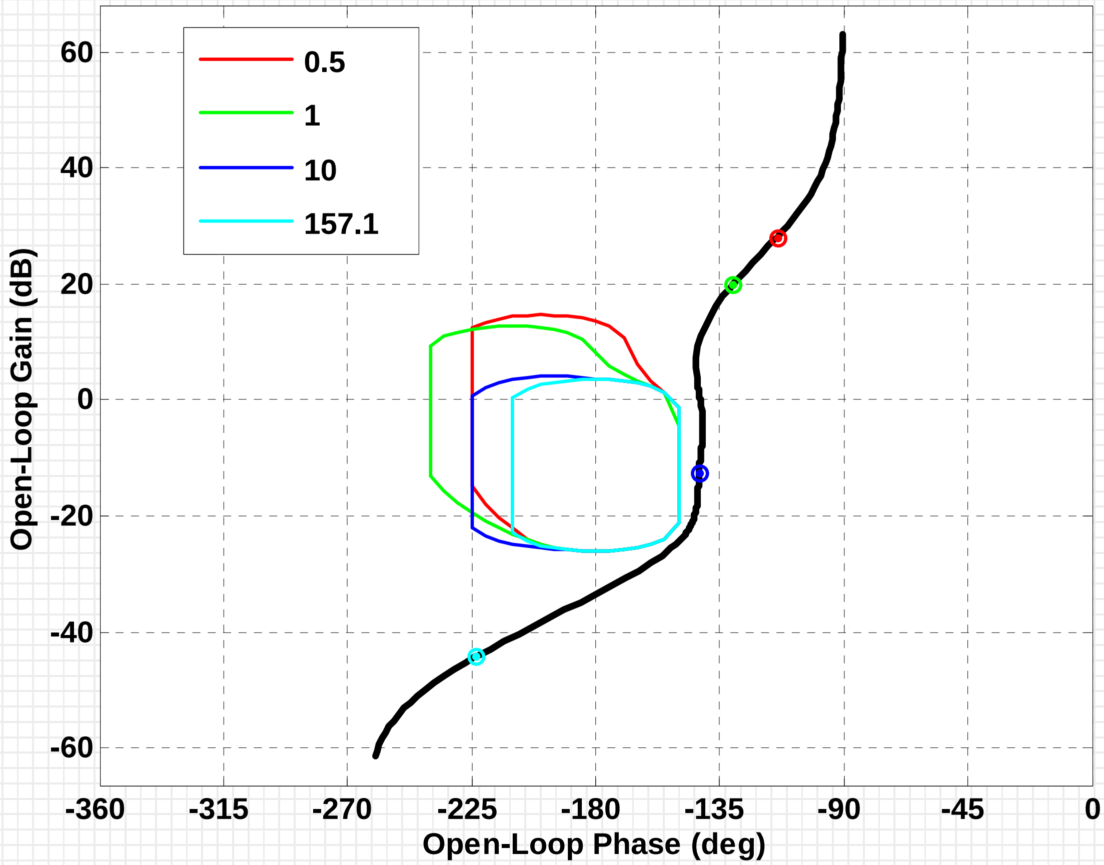
We conclude that the sampling time is too large, resulting in too much phase lag due to the NMP zero which rules out 20dB loop gain at $\omega_w = 1$. The specs cannot be achieved. A failed loop shaping attempt is shown next.

Such a large phase lag over a relatively small freq range indicates “max” effect of NMP zero which cannot be overcome with loop shaping



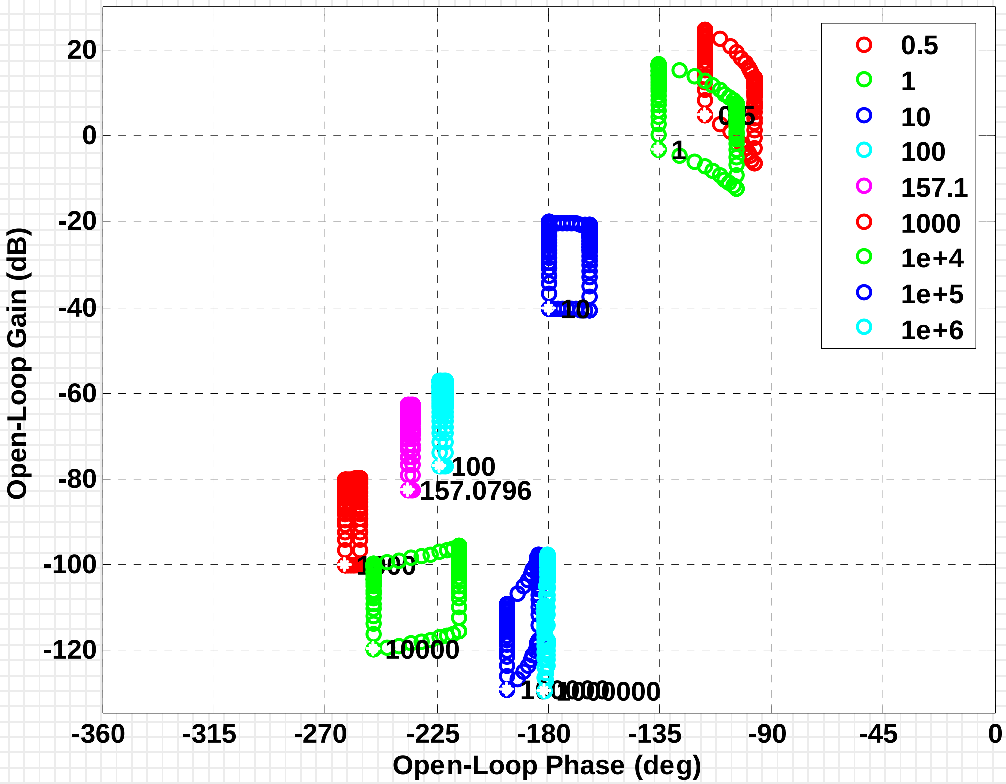
So we have to decrease the sampling time. How much?

To be on the safe side, we seek sampling time such that the all-pass's lag of 150° is at, say, above 300 r/s. For example, for a T_s = 0.02 sec, that frequency is about 360 r/s. A successful, stable design is shown below (`ch12_example.m` and `ch12_ex_a.shp`).



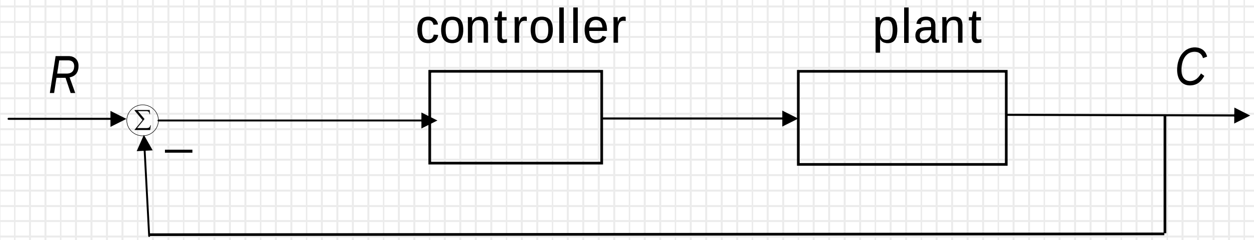
There's one trap we may have ignored. Recall that as $\omega \rightarrow \omega_s/2$, then $\omega_w \rightarrow \infty$. The dynamics of $P_{ZOH}(w)$ can occur at high frequencies; indeed, the nominal plant has a zero at 2400. Let us re-plot the templates, this time at higher frequencies.

We observe enlarging templates at, e.g., 10000 r/s!



12.3. Direct z-Plane Design

In the z domain, the block diagram is



and the specs are

$$\left|\frac{C}{R}(z)\right| = \left|\frac{P_{ZOH}(z)G(z)}{1+P_{ZOH}(z)G(z)}\right| \le 2,\ z = e^{jT_s\omega}, \omega \in \left[0, \omega_s/2\right], \forall P \in \mathcal{P}$$

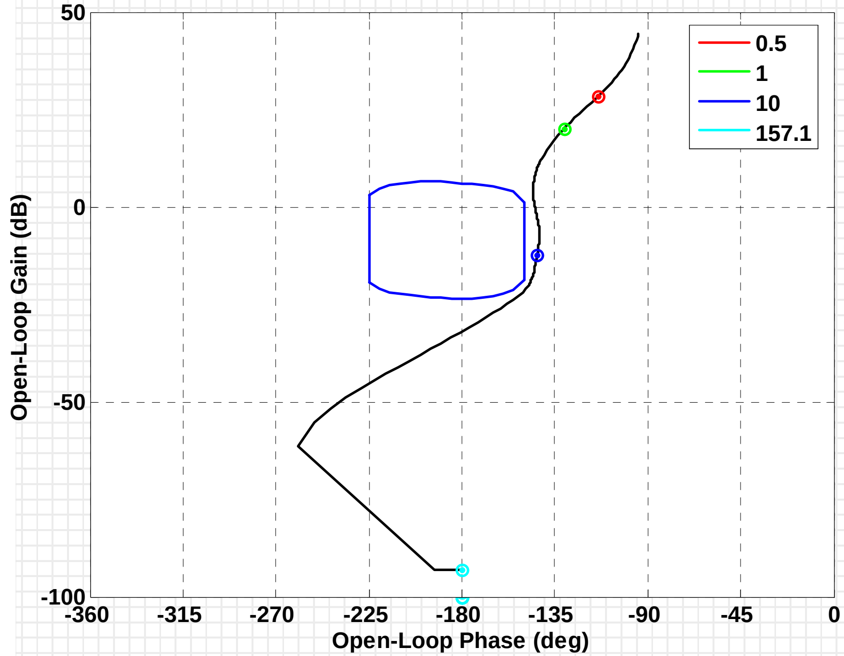
$$|L(z)| \ge 10,\ \ z = e^{jT_s 1}.$$

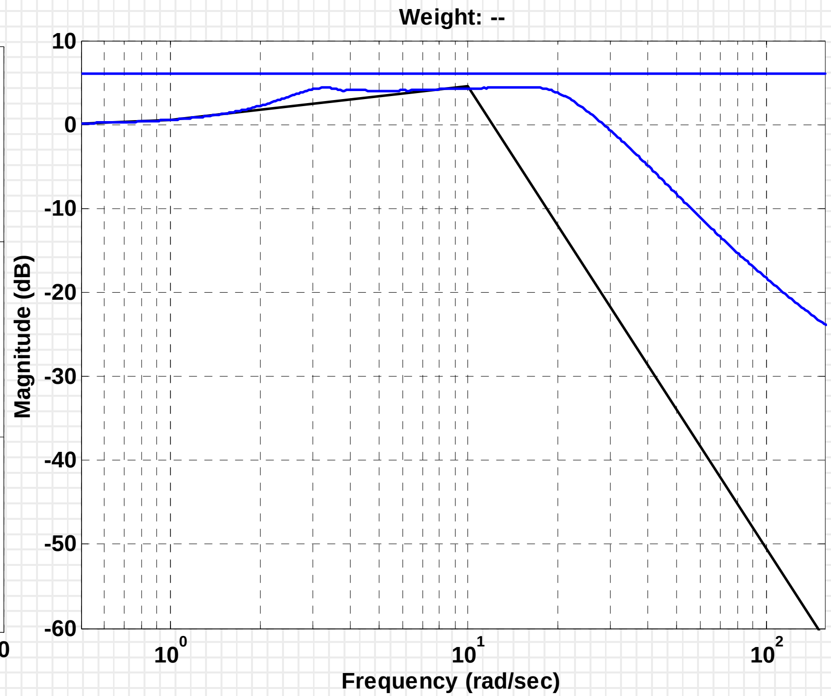
Note that frequency dependant weights can be converted using

A z-domain design is executed just like an s-domain design, but one must realize the z-domain frequency response deviates from its s-domain equivalence at we approach the Nyquist frequency.

Chapter 6 in the QFT Toolbox manual describes details of the LPSHAPE GUI in a discrete-time mode. A portion of this manual in a pdf version is discussed next.

A successful, stable design is shown below (`ch12_example_z.m` and `ch12_ex_z.dsh`). A comparison of w and z domain analysis is shown below (right). Can you explain the difference?





Note that robust stability is established in a similar manner as done in the s domain (see Ch. 4 in manual).

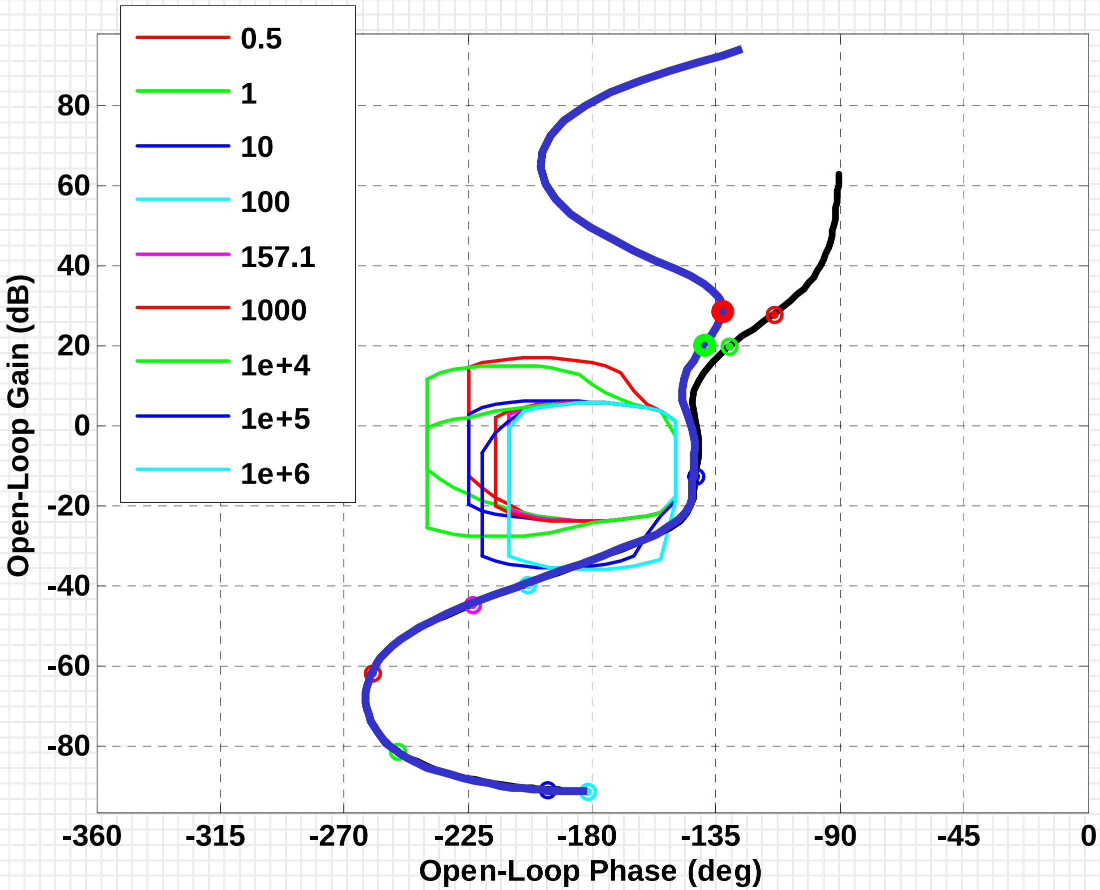
12.4. Phase Lag Maximization

It is possible to substantially increase low-frequency loop gain without modifying mid and high frequency responses. The idea is related to Bode gain-phase relation (IH book. CH. 10.3).

Consider two loops L_1 and L_2. It can be shown that ($\theta = \angle L$)

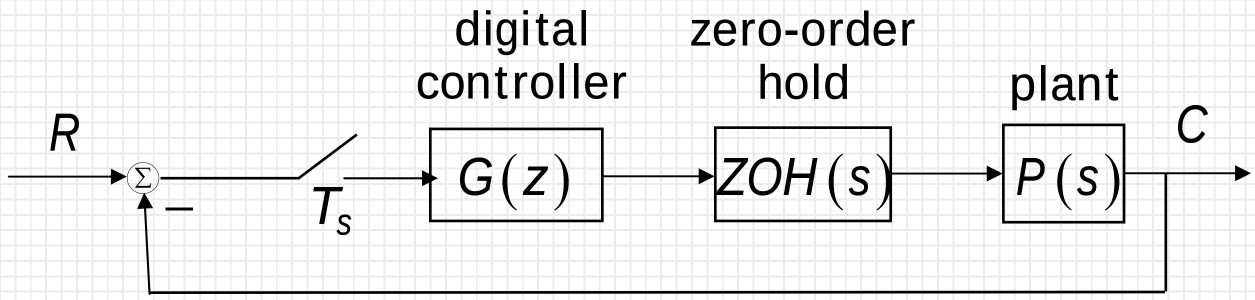
Similar tradeoff of phase lag (conditional stability here) can be exploited for other purposes, such as higher low-frequency gains. Note that conditional stability is not a viable solution in some applications.

Consider again our design earlier in this chapter (ch12_example.m and ch12_ex_a.shp). Adding phase lag at low- freq (via complex zeros then complex poles) allows for higher loop gains in that freq range (ch12_ex_b.shp). What is the tradeoff? Conditional stability.



12.5. Homework

Consider a sampled-data control system shown below



The plant is

$$\mathcal{P} = \left\{ \frac{k}{(s+a)(s+b)} : \quad k \in [1,10],\ a \in [1,5],\ b \in [20,30] \right\}.$$

The specs involve a margin constraint

$$\left| \frac{P_{ZOH}G}{1+P_{ZOH}G}(z) \right| \le 1.2, \text{ for all } P \in \mathcal{P},\ z = e^{j\omega T_s},\ \omega \in \left[0, \frac{\pi}{T_s}\right]$$

plant output disturbance rejection according to

$$\left| \frac{1}{1+P_{ZOH}G}(z) \right| \le \left| 0.02 \frac{z^3 + 64z^2 + 748z + 2400}{z^2 + 14.4z + 169} \right|,\ \forall P \in \mathcal{P},\ z = e^{j\omega T_s},\ \omega \in [0,10]$$

and plant input disturbance rejection according too

$$\left|\frac{P_{ZOH}}{1+P_{ZOH}G}(z)\right| \leq 0.01, \ \ \forall P \in \mathcal{P}, \ \ z = e^{j\omega T_s}, \ \ \omega \in [0,50].$$

Select a "maximal" sampling time such that the specs are met and we have robust stability. You should use the NMP zero analysis in the w domain to estimate best sampling time. You can execute either w or z domain design (or both!). <u>Show all pertinent work and derivations.</u>