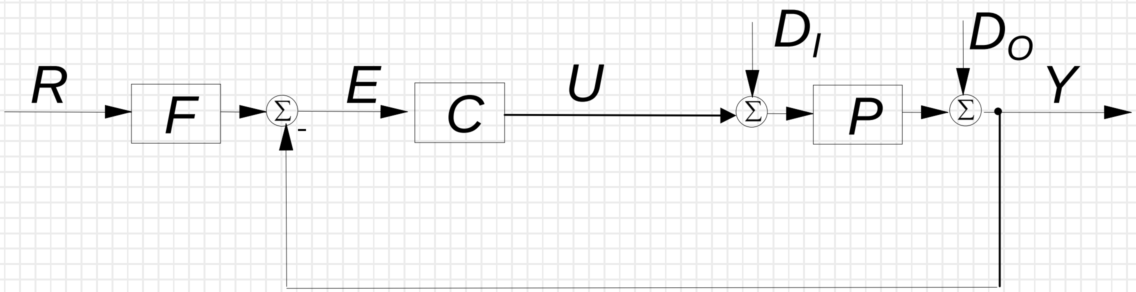
11. Discrete-Time Control: Background[1,2]

Discrete-time control system are hybrid: part continuous time and part continuous time.



In studying how to analyze such systems we'll visit:

- Impulse sampling and zero-order hold
- z transform
- Stability
- Design

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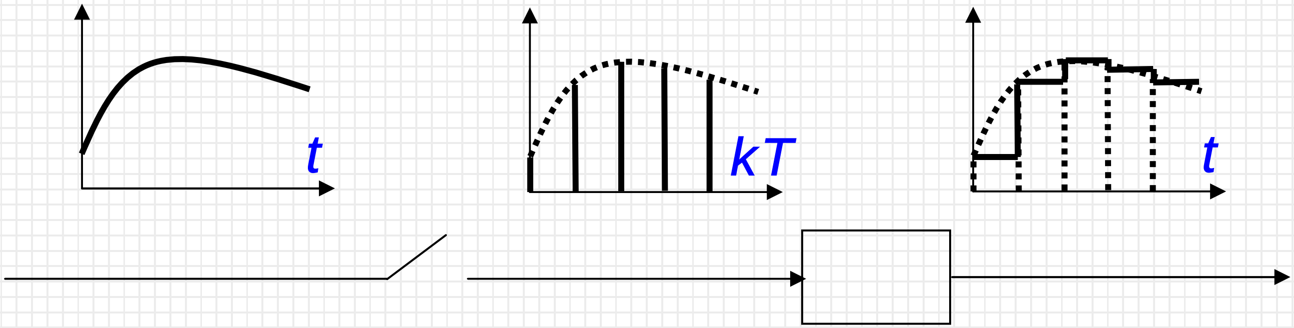
1. Ogata, K., *Discrete-Time Control Systems*, Prentice-Hall, Inc., 1987.
2. Houpis, CH., and Rasmussen, SJ., *Quantitative Feedback Theory Fundamentals and Applications*, Marcel Dekker AG, 1999.

11.1. Sampling and Hold

An ideal sampler comprises of a switch that closes to admit an input signal every sampling period *T*. The finite duration of the sampling is assumed infinitesimal. This is called *analog-to-digital* (A/D) conversion. A *digital-to-analog* (A/D) conversion converts to sampled-data signal back to a continuous-time signal – typically via a *zero-order hold* (ZOH) circuit.



For a signal zero at $t < 0$, $h(t)$ is related to $x(t)$ as follows

$$h(t) = x(0)\left[1(t) - 1(t-T)\right] + x(T)\left[1(t-T) - 1(t-2T)\right] + x(2T)\left[1(t-2T) - 1(t-3T)\right] + \ldots$$

The Laplace transform of a delayed step is

and for the ZOH we jave

$$\mathcal{L}\left[h(t)\right] = H(s) = \sum_{k=0}^{\infty} x(kT)\frac{e^{-kTs} - e^{-(k+1)Ts}}{s}$$

We define

The sampled signal is a train of impulses (the strength of each equals $x(t)$ at kT)

where $\delta(t-kT)=0$ unless $t=kT$.

Nyquist (Shannon's) sampling theorem says:

A function $f(t)$ which contains no frequency components greater than ω_c (i.e., band limited) can be represented by $x(kT)$ with $T<\pi/\omega_c$.

In addition to assuming no *aliasing*, we assume that *quantization*, *saturation*, and *conversion* errors in the A/D conversion are negligible.

11.2. The z Transform

Define

so we can write

The machinery developed for Laplace transform domain, such frequency response, root locus, and stability analysis is readily applicable in the z domain for discrete-time control system having sampling as discussed above.

Examples of z transforms:

$$x(t)=\delta(0)=\begin{cases}1, & k=0\\ 0, & k\neq 0\end{cases}$$

so

The unit step function

$$x(t)=1(t)$$

has a z transform of

A table of z transforms is provided.

Important Properties.

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- *Final Value Theorem*: if $X(z)$ has only stable poles (inside the unit circle) with the exception of one at $z = 1$, then the final value is

A table of important z transform properties is provided.

Inverse z transform.

- direct division
- partial fraction expansion
- other methods

Example. Find $x(k)$ for $X(z)=\dfrac{10z+5}{(z-1)(z-.2)}=\dfrac{10z^{-1}+5z^{-2}}{1-1.2z^{-1}+0.2z^{-2}}$.

$$
\begin{array}{r|l}
 & 10z^{-1}+17z^{-2}+18.4z^{-3}+16.86z^{-4}+\cdots \\
\hline
1-1.2z^{-1}+0.2z^{-2} & 10z^{-1}+5z^{-2} \\
 & 10z^{-1}-12z^{-2}+2z^{-3} \\
\cline{2-2}
 & 17z^{-2}-2z^{-3} \\
 & 17z^{-2}-20.4z^{-3}+3.4z^{-4} \\
\cline{2-2}
 & 18.4z^{-3}-3.4z^{-4} \\
 & 18.4z^{-3}-22.08z^{-4}+3.68z^{-5} \\
\cline{2-2}
 & 18.68z^{-4}-3.68z^{-5} \\
 & 18.68z^{-4}-22.416z^{-5}+3.736z^{-6} \\
\cline{2-2}
\end{array}
$$

Since

$$X(z) = \sum_{k=0}^{\infty} x(kT) z^{-k},$$

direct comparison of this infinite series with the long division gives

Example. Find $x(k)$ for $X(z) = \dfrac{10z}{(z-1)(z-.2)}$

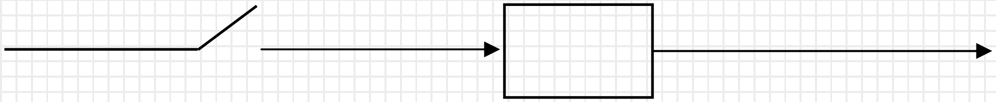
If $X(z)$ has one or more zeros at the origin, use this trick

From z transform tables

Note: if we did not divide by z, the expansion of $x(z)$ would yield terms not appearing in the table.

11.3. The Pulse Transfer Function

Here we study transfer functions in the z domain. Consider the system shown below.



The continuous-time convolution integral becomes a convolution integral in discrete time

$$y(kT) = \sum_{h=0}^{\infty} g(kT - T)x(hT)$$

$$= \sum_{h=0}^{\infty} x(kT - hT)g(hT), \;\; x(k <)h = g(k < h) = 0.$$

The z transform of $y(kT)$ is

$$Y(z) = \sum_{k=0}^{\infty} y(kT)z^{-k} = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} g(kT - hT)x(hT)$$

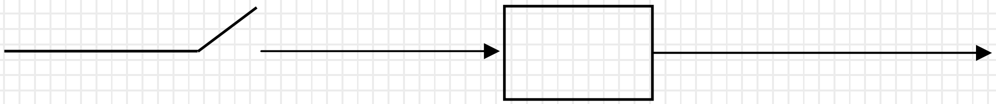
$$= \sum_{m=0}^{\infty} \sum_{h=0}^{\infty} g(mT)x(hT)z^{-(m+h)} = \sum_{m=0}^{\infty} g(mT)z^{-m} \sum_{h=0}^{\infty} x(hT)z^{-h}$$

$$= G(z)X(z), \quad m = k - h.$$

The pulse transfer function is

$$G = \frac{Y(z)}{X(z)}.$$

Back to the original system.



To relate sampled signals, we note that

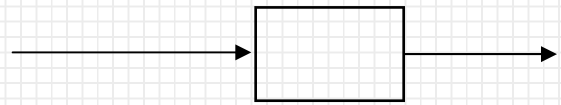
$$X^*(s) = X^*(s \pm j\omega_s k), \quad k = 0, 1, 2, \ldots$$

so

and the starred Laplace transform with $z = e^{sT}$ becomes

$$Y(z) = G(z)X(z).$$

Note that in this system

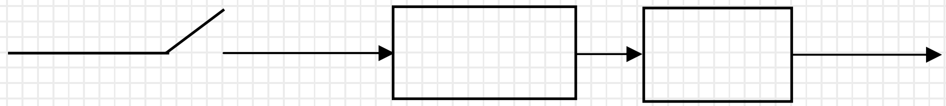


That is

How do we obtain a pulse TF for a system? Typically, we use basic block diagram algebra and z-Transform Tables (see attachments).

Example. Find the PTF of

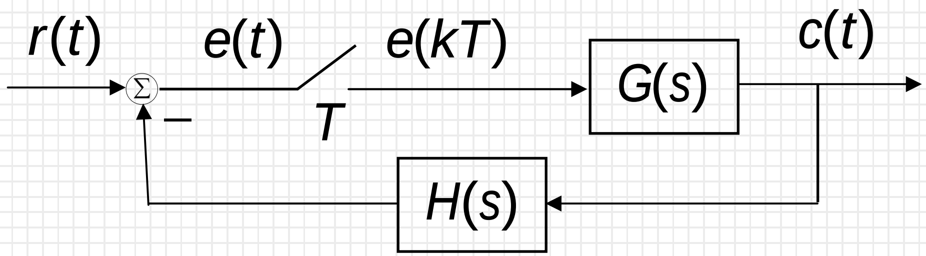
$$G(s) = \frac{1-e^{-Ts}}{s}\frac{1}{s(s+1)}$$



$$G(z)=\mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\frac{1}{s(s+1)}\right]=\left(1-z^{-1}\right)\mathcal{Z}\left[\frac{1}{s^2(s+1)}\right]=\left(1-z^{-1}\right)\mathcal{Z}\left[\frac{1}{s^2}+\frac{1}{s}+\frac{1}{s+1}\right]$$

$$=\left(1-z^{-1}\right)\left[\frac{Tz}{\left(1-z^{-1}\right)^2}+\frac{1}{1-z^{-1}}+\frac{1}{1-e^{-T}z^{-1}}\right]=\frac{\left(T-1+e^{-T}\right)z^{-1}+\left(1-e^{-T}-Te^{-T}\right)z^{-2}}{\left(1-z^{-1}\right)\left(1-e^{-T}z^{-1}\right)}.$$

Example. Find the PTF of a closed-loop discrete-time control system shown below.



$$E=R-HC=R-HGE^*$$

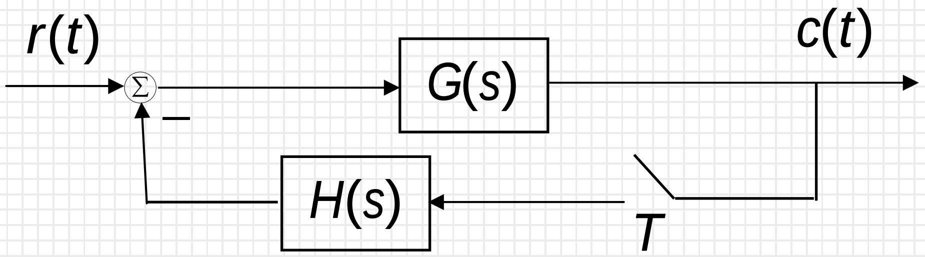
Starring both sides

$$\frac{E^*}{R^*}=\frac{1}{1+[HG]^*}$$

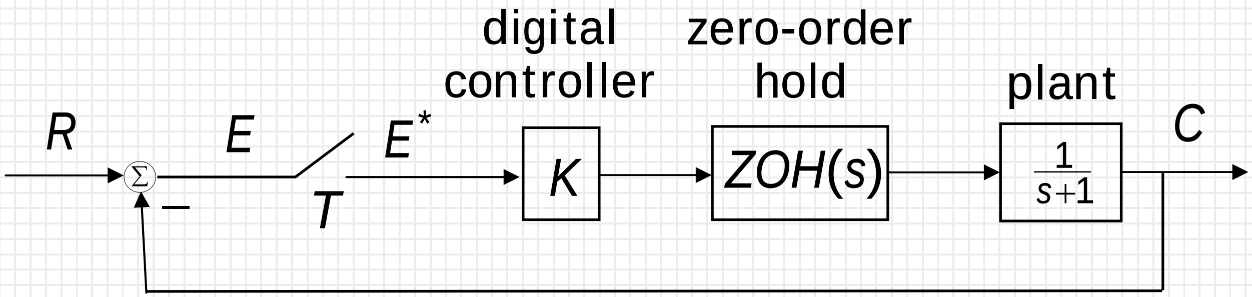
Combining the last 2 equations gives

Note: we have assumed all samples have same sampling period and are synchronized.

Some systems do not have a PTF. This occurs when the input signal dynamics cannot be decoupled from the dynamics of the system.



Example.



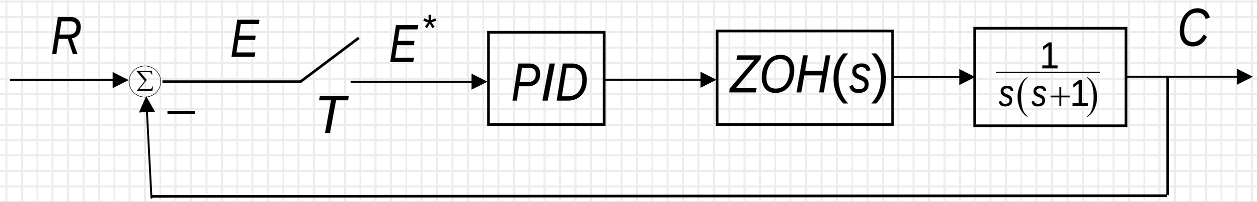
$$C = kGE^*$$

$$E = R - C$$

$$= \left(1-z^{-1}\right)\mathcal{Z}\left[\frac{k}{s}-\frac{k}{s+1}\right] = \left(1-z^{-1}\right)\left[\frac{k}{1-z^{-1}}-\frac{k}{1-e^{-T}z^{-1}}\right] = \frac{k\left(1-e^{-T}\right)z^{-1}}{1-e^{-T}z^{-1}}$$

$$\Rightarrow \frac{C}{R}(z) = \frac{k\left(1-e^{-T}\right)z^{-1}}{1-\left[k-(k+1)e^{-T}\right]z^{-1}}.$$

Example.

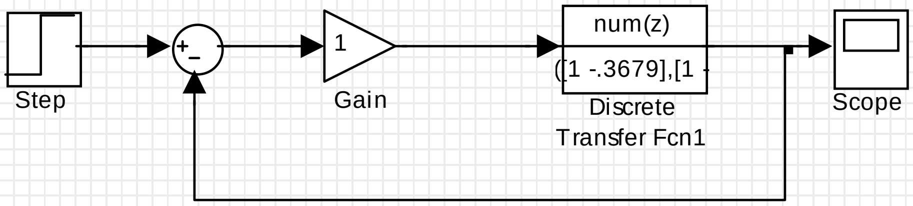


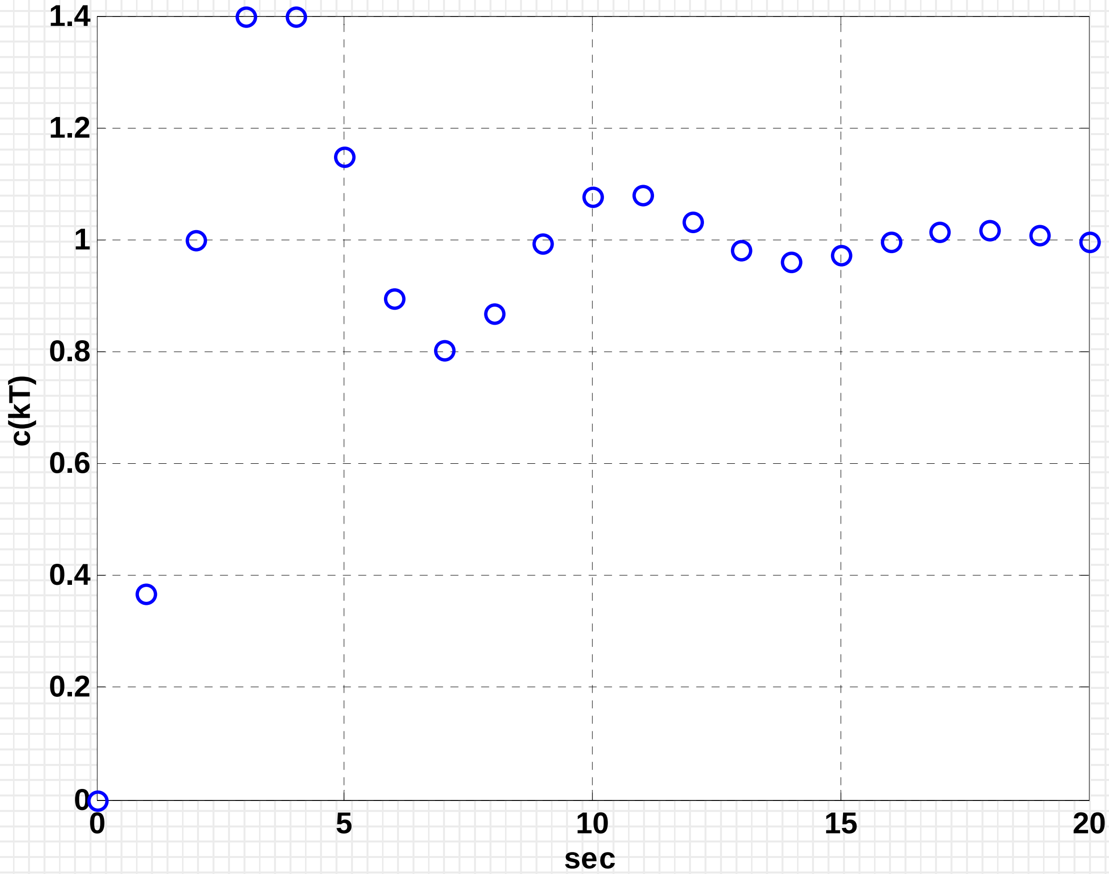
$$PID(s) = K_p + K_i^{-1}\tfrac{1}{s} + K_d s \quad \Rightarrow \quad PID(z) = K_p + \frac{T}{K_i}\frac{1}{1-z^{-1}} + \frac{K_d}{T}\left(1-z^{-1}\right)$$

$$G(z) = \mathcal{Z}\left[ZOH(s)G(s)\right] = \left(1-z^{-1}\right)\mathcal{Z}\left[\frac{1}{s^2(s+1)}\right] = \frac{.3697z^{-1} + .2642z^{-2}}{\left(1-.3697z^{-1}\right)\left(1-z^{-1}\right)}.$$

Let $K_d = K_i^{-1} = 0, K_p = 1.$

$$\frac{C}{R}(z) = \frac{K_pG(z)}{1+K_pG(z)} = \frac{.3697z^{-1} + .2642z^{-2}}{1-z^{-1}+.6321z^{-2}} = \frac{.3697z^{1} + .2642}{z^2 - z^1 + .6321}.$$


Step
1
Gain
num(z)
([1 -.3679],[1 -
Discrete
Transfer Fcn1
Scope


1.4
1.2
1
0.8
0.6
0.4
0.2
0
c(kT)
0
5
10
15
20
sec

For a unit step

$$C(z)=\frac{.3697z^{-1}+.2642z^{-2}}{1-z^{-1}+.6321z^{-2}}\frac{1}{1-z^{-1}}$$
$$=.3679z^{-1}+z^{-2}+1.3996z^{-3}+1.3996z^{-4}+1.1469z^{-5}+.8944z^{-6}+\ldots$$

Since z^{-1} implies time shift by one sampling period, taking inverse z transform gives

Note: it is possible to compute the response between sampling instances.

11.4. s-plane to z-plane Mapping

Stability and performance of continuous-time (CT) systems depends on pole location. Since s and z are related by $z = e^{Ts}$, we can study these discrete-time properties of a PTF by related z domain pole location via the map.

Poles and zeros in the s plane are mapped to the z plane via $z = e^{Ts}$. Denote the complex number $s = \sigma + j\omega$ so

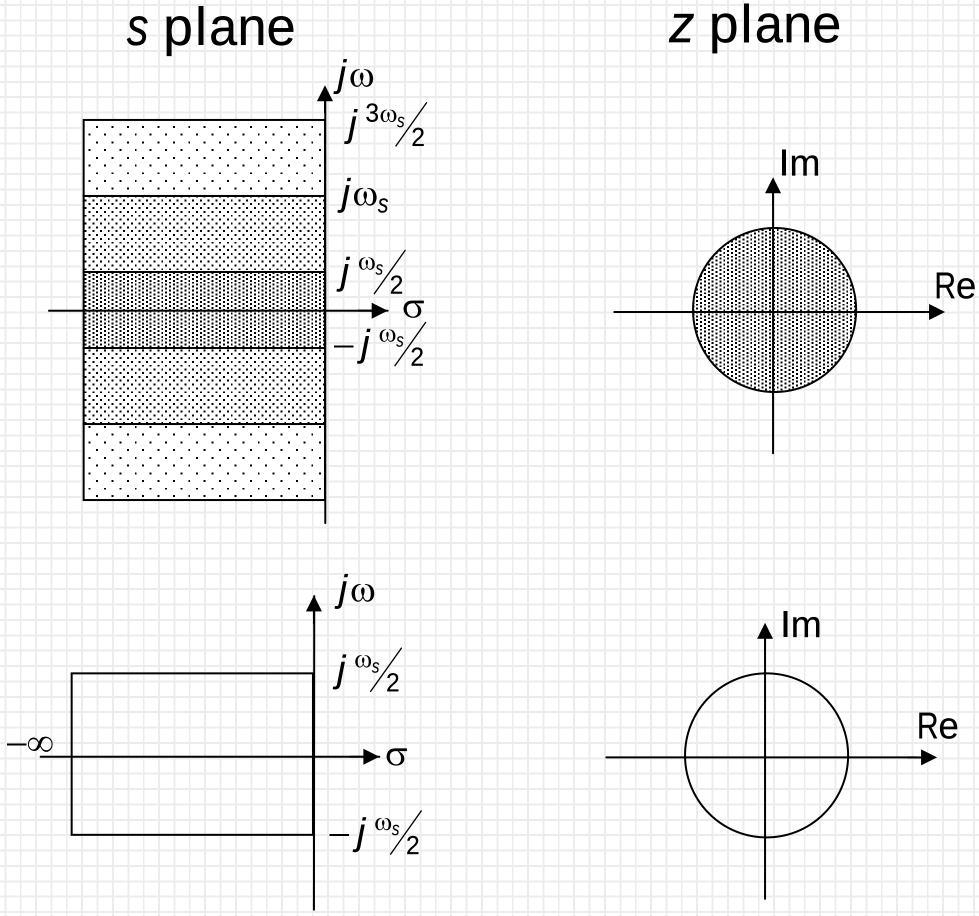
$$z = e^{T(\sigma + j\omega)} = e^{T\sigma} e^{jT\omega} = e^{T\sigma} e^{j(T\omega + 2\pi k)}, k = 0, \pm 1, \ldots$$

Note that s plane frequencies with integer multiple of ω_s difference are mapped into the same z plane location.

Stable systems have al their poles in the open left-half s plane, or in the z plane

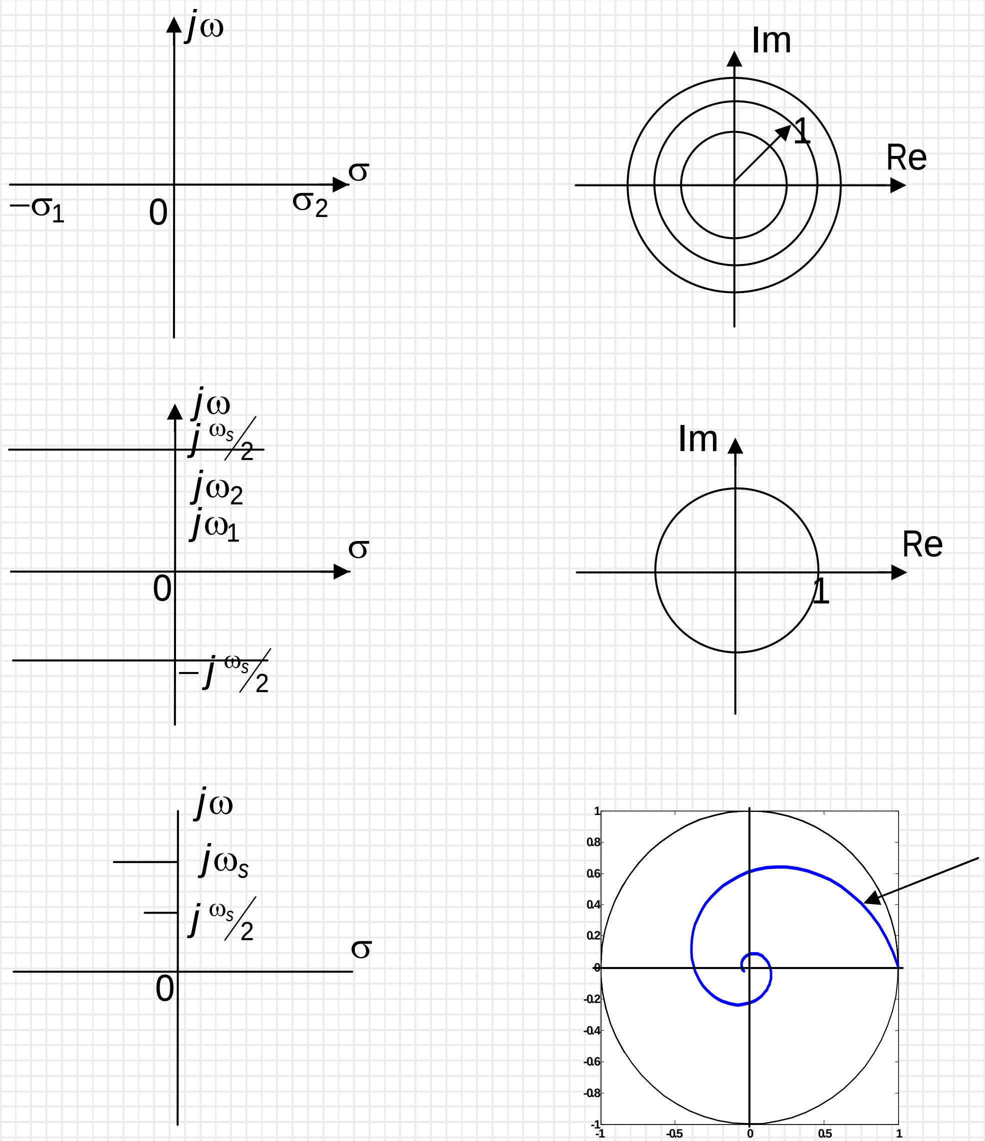
The $j\omega$-axis in the s plane maps into $|z| = 1$ circle.

The phase of z, ωT, varies from $-\infty$ to ∞ as ω varies from $-\infty$ to ∞. In particular, along the $j\omega$ axis, as ω varies from $-\frac{1}{2}\omega_s$ to $-\frac{1}{2}\omega_s$ we have $|z|=1$ and $\angle z$ varies CCW from $-\pi$ to π. From $-\frac{1}{2}\omega_s$ to $-\frac{1}{2}\omega_s$ the phase $\angle z$ varies again CCW from $-\pi$ to π. This is depicted below.

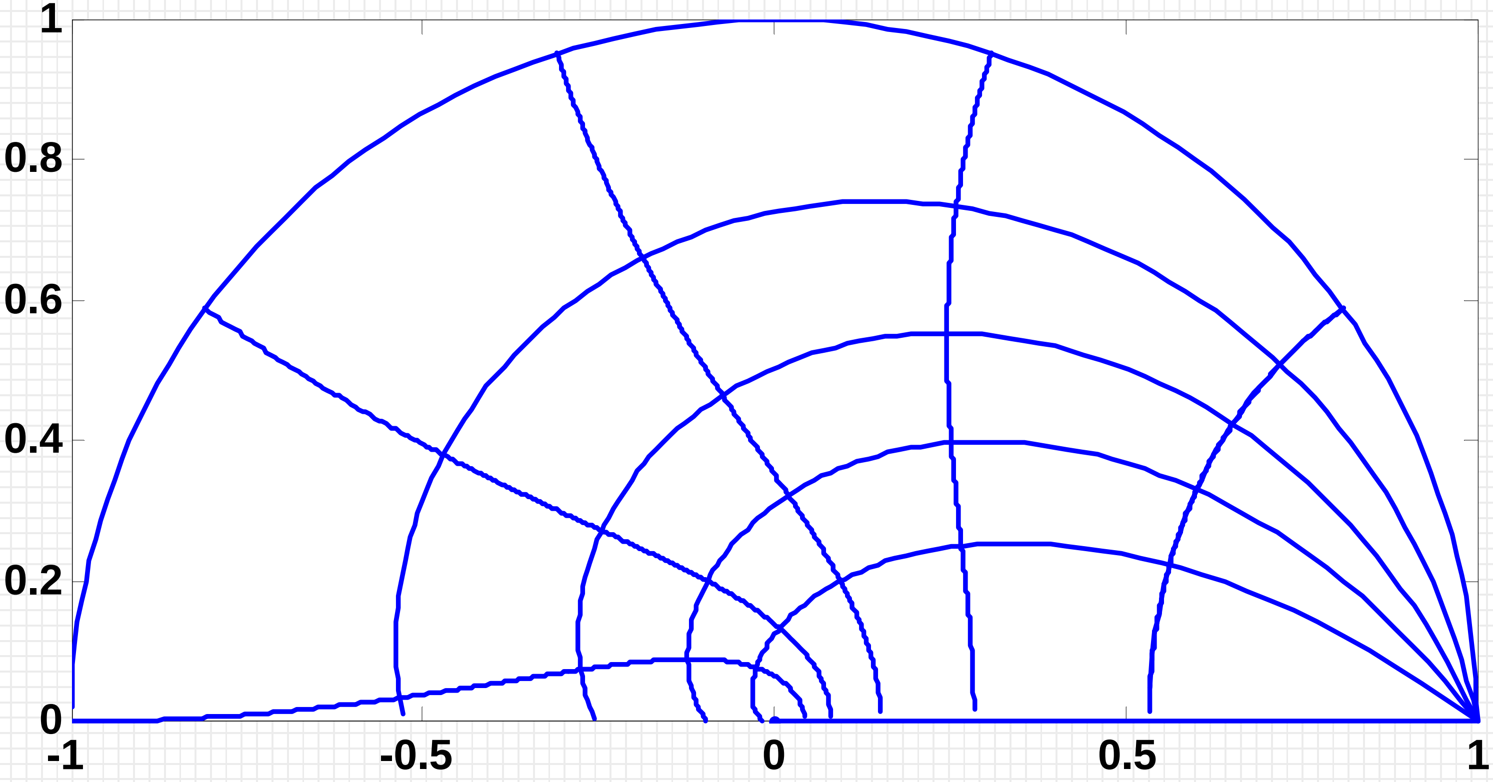


primary

s plane / *z* plane



z plane

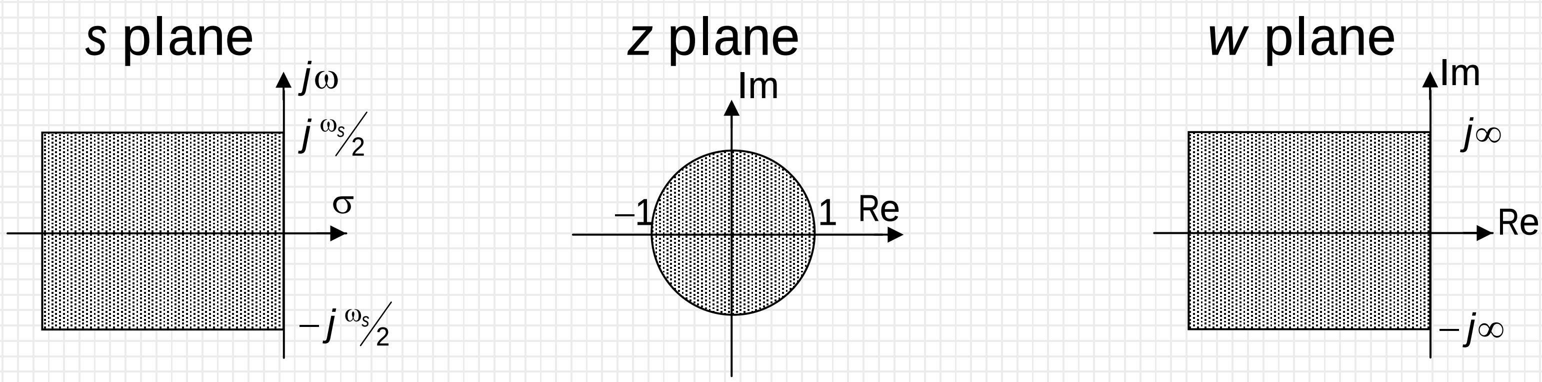


11.4. Stability Analysis

Discrete-time stability is determined from the roots of the characteristic equation $1+L(z)=0$.

11.4.1 Stability tests

- Jury Stability Test an algebraic test using the characteristic equation.
- Routh test via bilinear transformation.



$$0 = 1 + L(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z^1 + a_0$$

$$a_n \left(\frac{w+1}{-w+1} \right)^n + a_{n-1} \left(\frac{w+1}{-w+1} \right)^{n-1} + \cdots + a_1 \left(\frac{w+1}{-w+1} \right)^1 + a_0 = 0$$