10. NON-MINIMUM PHASE ZEROS

While the design procedure for plants with non-minimum phase (NMP) zeros is the same as for all other plants, the system itself faces certain limitations. We begin by observing such limitations in a SISO system using a design example (*Horowitz*, Section 8.4), then study these theoretically, and finally, generalize to MIMO systems.

Assume $P(s)$ has NMP zeros and factor is as follows

where

Denote the nominal plant by

The loop transmission is

Where $A_0(s)$ is an all-pass function ($|A_0|=1|$) and $L_{m0}(s)$ is minimum phase (assuming C is so). $V(s)$ is the only uncertain part in $L(s)$

Also,

Let us proceed with an illustrative example. The plant family is described by

$$\mathcal{P} = \left\{ P = \frac{k(1-\tau s)}{s(1+\beta s)} \ : k \in [1,3],\ \beta \in [0.3,1],\ \tau \in [0.05,1] \right\}.$$

There is only one spec here

$$|T(j\omega)| \le 1.4 \ (3\ dB),\ \omega \ge 0,\ \text{for all } P \in \mathcal{P}.$$

Using our notation, and arbitrarily assuming nominal values

so

Recall

$$L(s) = \frac{N_0(s)}{N_0(-s)} P_{10}(s) N_0(-s) C(s) \frac{N(s)P_1(s)}{N_0(s)P_{10}(s)}$$

$$= A_0(s) L_{m0}(s) V(s)$$

so after simplifications we get

Finally, evaluating the above numerically gives

Again

$$L(s) = A_0(s) L_{m0}(s) V(s)$$

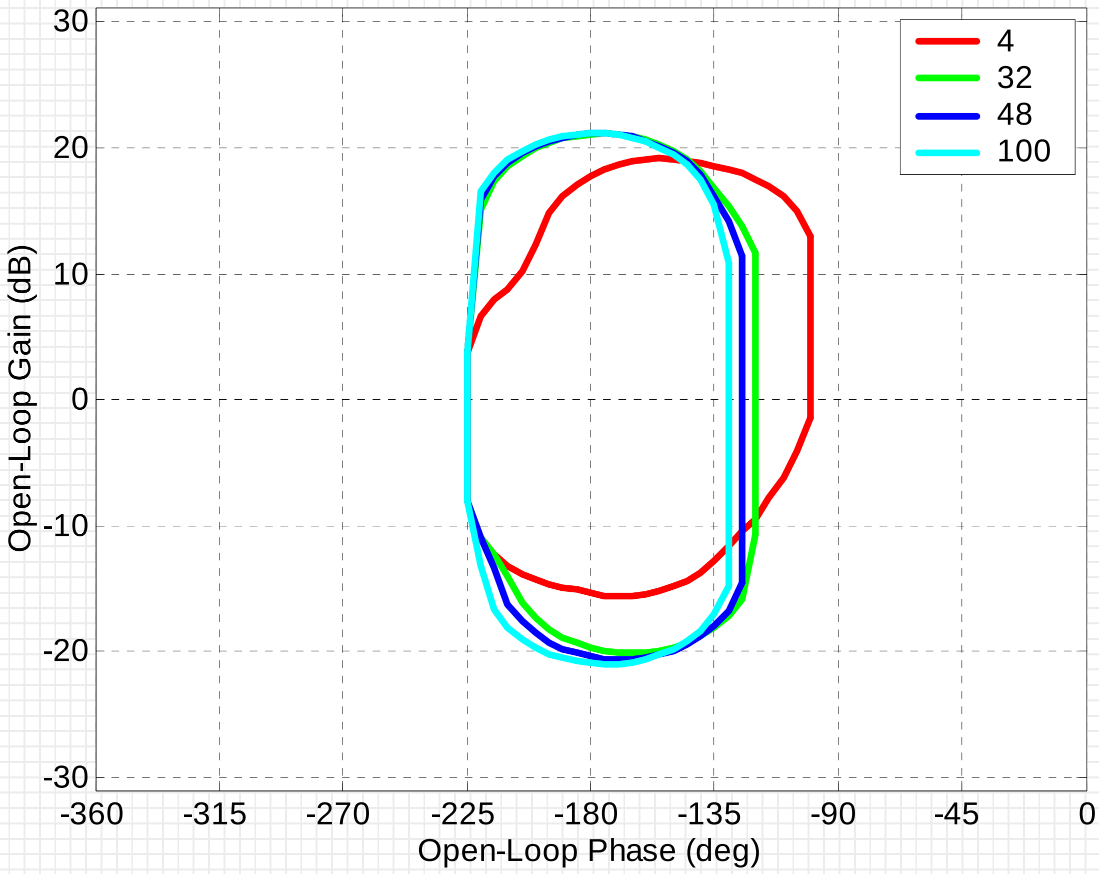
so

with

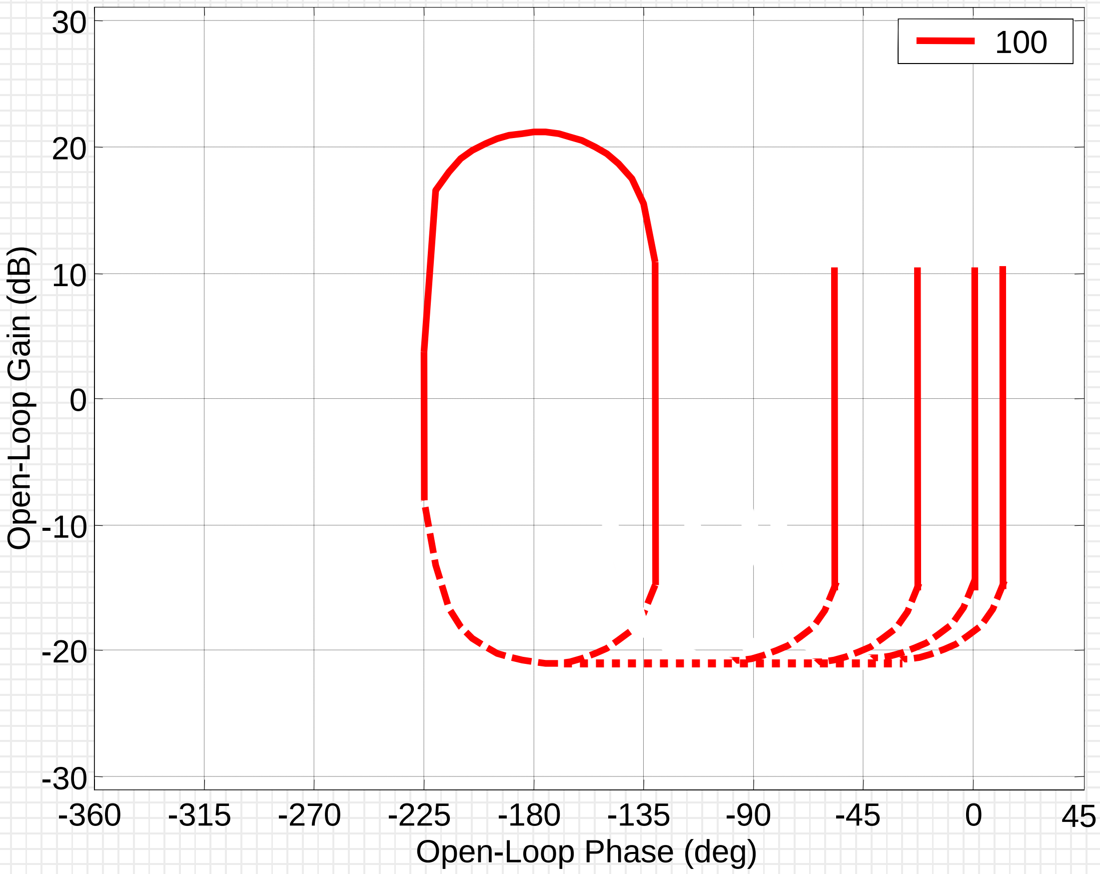
Denote the bound on $L(j\omega)$ by $B_n(\omega)$. It follows that the bound on $L_{m0}(j\omega)$ is $B_m(\omega)$ where

That is, $B_m(\omega)$ is simply $B_n(\omega)$ shifted to the right (on an NC) by the phase lag of $A_0(j\omega)$.

The margin bounds are shown below.



For discussion purposes, we assume all bounds are the same as the one at 100. In addition, we assume some low-frequency performance bound at $\omega = 4$ as shown below.



Note that $\angle A_0 = -77^\circ @ \omega = 16,\ -115^\circ @ \omega = 32, -130^\circ @ \omega = 43,$

$$-134^\circ @ \omega = 48, -180^\circ @ \omega = \infty.$$

Using Bode gain-phase relations, it can be shown that there is no $L_{m0}(j\omega)$ satisfying its bound at $\omega = 4$ and its margin bounds. The reason is as follows.

Suppose $L_{m0}(j4)$ is at [20dB,-90°]. In the crossover range, the average loop slope cannot be larger than –26 dB/dec, implying an average phase of -120°. So $L_{m0}(j48)$ is approx. (26+9) dB below $L_{m0}(j4)$. Specifically, it is at [-15dB,-150°].

However, $B_m(48)$ requires the loop to have positive phase. Hence, no practical L_{m0} can be designed to synthesized these bounds (see also *Sidi*, Section 4.5).

If L_{m0} does not reach the point x at $\omega = 43$, then it is impossible to complete the design without relaxing the specs or reducing uncertainty. In essence, positive loop phase in [43,100] implies an increase of $|L_{m0}|$ in that range. This is in contrast to the required mag reduction at increasing frequencies.

We conclude that NMP zeros result in a hard limitation on loop bandwidth. This conclusion is invariant to the choice of nominal plant.

A similar limitation is seen with a pure time delay

$$e^{-\tau s}.$$

This can be seen from its Pade (series) approximations

$$e^{-\tau s} \approx \frac{1-\frac{\tau s}{2}+\frac{(\tau s)^2}{8}-\frac{(\tau s)^3}{48}+\cdots}{1+\frac{\tau s}{2}+\frac{(\tau s)^2}{8}+\frac{(\tau s)^3}{48}+\cdots}.$$

Depending on the frequency range of interest, a 1st-order approximation often suffices

Hence, at best, the limitation of a pure time delay can be assumed to be that of an all-pass filter. The more terms needed, the worse the limitation. The larger the time delay, the smaller the value of the NMP zero, and the worse the limitations.

In practice, it is convenient to loop shape with the actual NMP loop L_{n0}. The above was meant for illustration purposes only.

10.1. SISO NMP LIMITATIONS

(Sidi 3.4., Yaniv 2.3.) Any stable transfer function can be uniquely factored into a stable, MP part L_m and an all-pass part A

$$L(s) = L_m(s)A(s).$$

Now assume that between the cross-over frequency ω_ϕ (i.e., $|L(j\omega_\phi)|$ = 0dB) and the gain-margin frequency ω_M we have

For a single RHP zero at a

$$L(s) = L_m(s)\frac{a-s}{a+s}$$

so

The following are useful relations between the NMP zero, phase margin ϕ, *PM* frequency ω_ϕ , and *GM* frequency ω_M

Also, the gain margin *GM* is

The above relations remain fairly accurate even when the assumption of asymptotic magnitude slope is somewhat violated.

The above relations can be extended to plants with multiple NMP zeros. If, within the frequency range of interest the NMP zeros z_i satisfy

$$\tan\left(\frac{\omega}{z_i}\right) \approx \frac{\omega}{z_i}$$

Then we can replace them with a single NMP zero z satisfying

This approximation holds only within this frequency range.

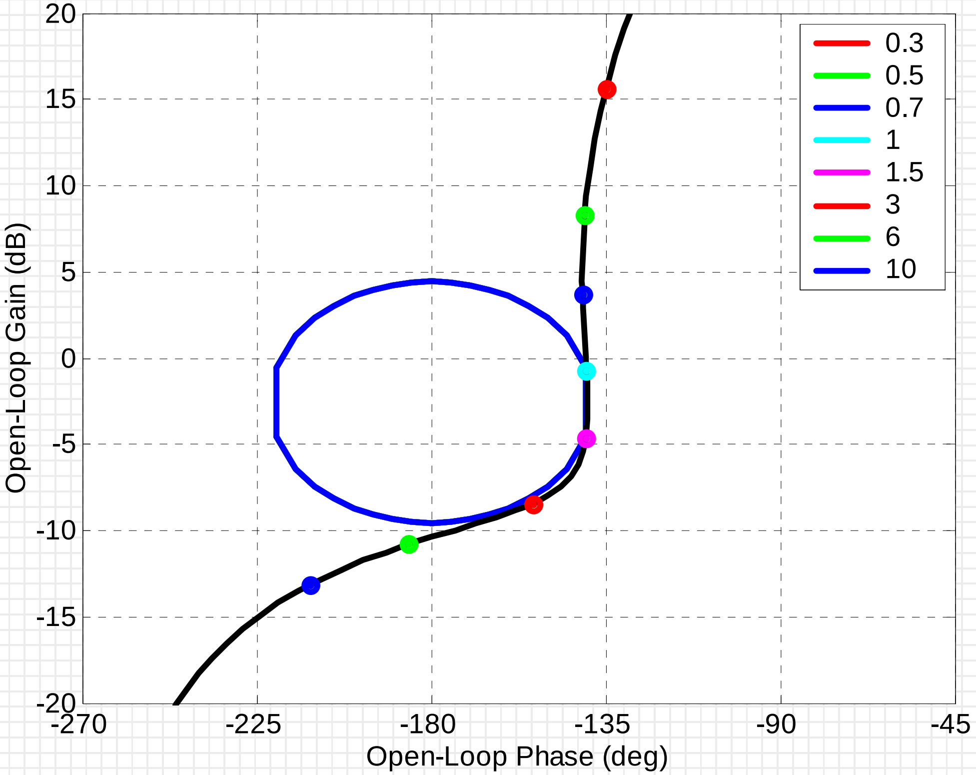
Example. Consider a plant with a single NMP zero

$$P = \frac{(3-s)}{s(8+s)}.$$

For phase margin ϕ = 40° and GM = 10 dB, the above relations give

$$\alpha = 0.61,\ \ \omega_{a\phi} = 0.275,\ \ \omega_{\phi} = a\omega_{a\phi} = 0.825,\ \ \omega_{M} = 2.8.$$

Using loop-shaping to maximize cross-over frequency with the spec $|S| \leq 1.5$ is shown to the right.



The actual ω_ϕ is 0.93 which is larger by 12%. However, ω_M is 5.7 or about 150% larger. So while we have a slightly larger cross-over frequency, we suffer a much larger increase in high-frequency loop gains with the accompanying sensitivity to sensor noise at the plant input and to un-modeled resonances.

We conclude the discussion on NMP SISO system with the following result.

Let the open loop be factored as before $L = L_m A$. Suppose that there exist solution to $PM - \angle A(j\omega) = 180^\circ$ where $\omega 1$ ($\omega 2$) is the lowest (highest) frequency that solved the above equation. Then

Horowitz (see his book) shows how to achieve large loop gains at frequencies beyond this. Moreover, if the plant has several NMP zeros, he shows how to achieve large loop gains at several frequency ranges. Bottom line: for each NMP zero there is a frequency range with the open-loop gain below 0 dB.

10.2. Notes

An open-loop NMP zero will appear in the complimentary sensitivity function. For example, let

$$L = L_{mp}\,\frac{s-a}{s+a}$$

then

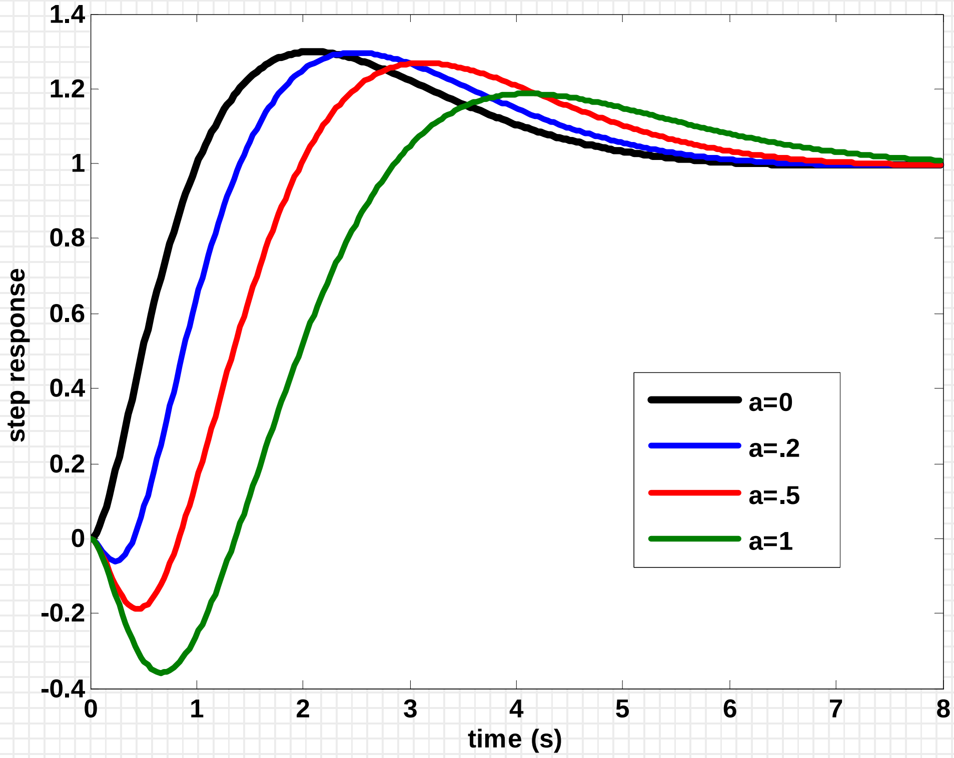
and we assume that there're no unstable pole zero cancellations in L.

NMP zeros introduce inevitable delays and undershoots in step responses. This is shown next.

Consider a complimentary sensitivity function

$$T(s) = \frac{1+2s}{(1+2s)\left(1+2\times .8s+s^2\right)}\frac{1-as}{1+as}.$$

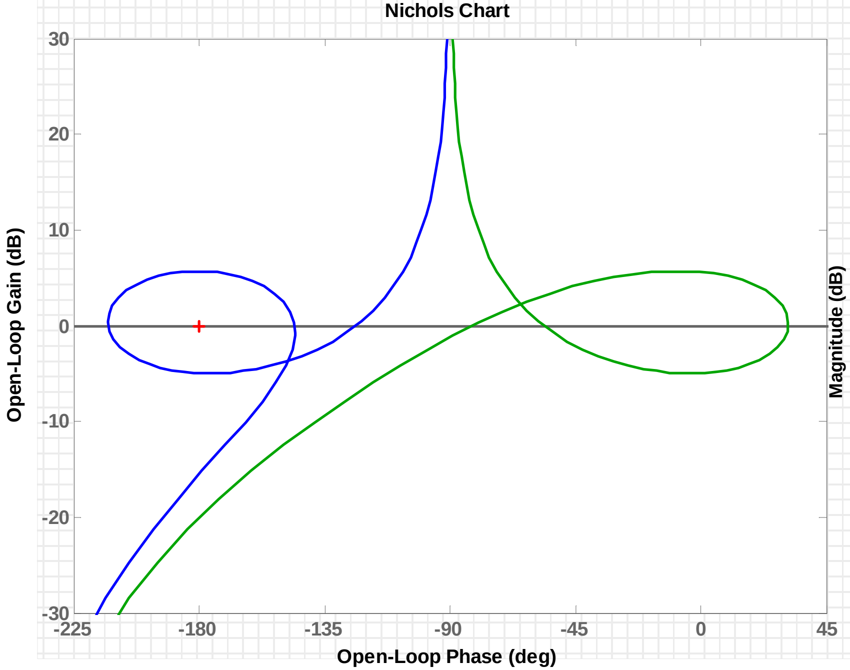
Various step responses are shown below.

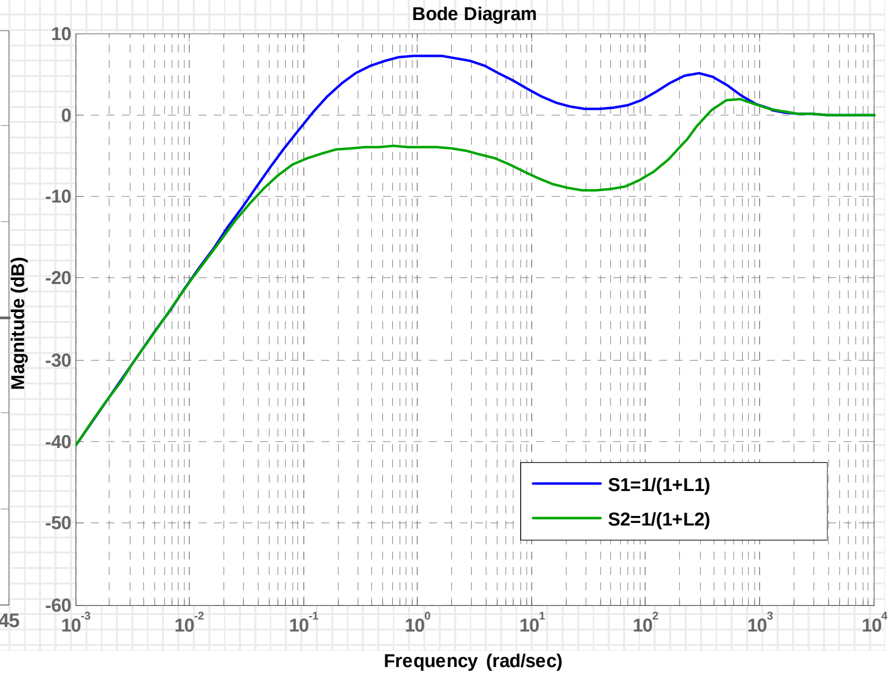


(Sidi 3.5.) Consider an open loop with both NMP zeros and unstable poles.

$$L_1(s) = \frac{1}{1.89}\frac{(s-.2)(s/3-1)}{s(s/12-1)(s/120-1)(s/1000+1)^2}$$

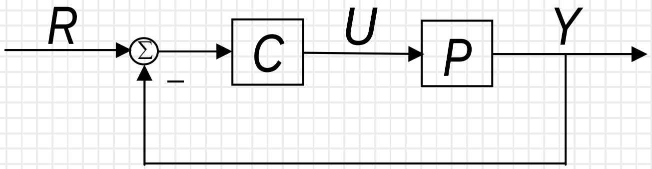
$$L_2(s) = \frac{1}{1.89}\frac{(s+.2)(s/3+1)}{s(s/12+1)(s/120+1)(s/1000+1)^2}$$





10.3. HOMEWORK

Consider the following feedback system



$$\mathcal{P} = \left\{ P(s) = \frac{k(1\text{-}ds)}{s(1+bs)} : \quad k \in [1,3],\ b \in [.3,1],\ d \in [0.05, 0.1] \right\}.$$

Using mag/phase relations from Chap. 8 and the discussion in this chapter, show that it is not possible to design a controller C such that it robustly achieves $|T| \le 3dB, \forall P \in \mathcal{P}$ and the tracking specs:

$$T_u(s) = \frac{1.28(s+2.2)(s+16.36)(s+30)}{(s+2.2)^2\left(s^2+2\times 0.9\times 3.5s+3.5^2\right)}$$

$$T_l(s) = \frac{122.5}{(s+10)\left(s^2+2\times 0.9\times 3.5s+3.5^2\right)}.$$

Next, relax the tracking spec to

$$T_u(s) = \frac{1.28(s+2.2)(s+16.36)(s+30)}{(s+2.2)^2\left(s^2+2\times 0.9\times 3.5s+3.5^2\right)}$$

$$T_l(s) = \frac{20}{(s+5)\left(s^2+2\times 0.9\times 3.5s+3.5^2\right)}$$

and attempt to execute a QFT design (relax $T_l(s)$ a bit further if necessary).

