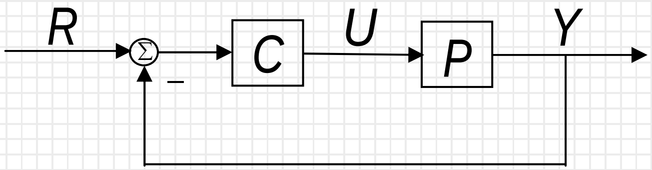
5. Computing QFT Bounds

Let us investigate the role of QFT bounds in designing control systems using a constructive example. Consider the following feedback system



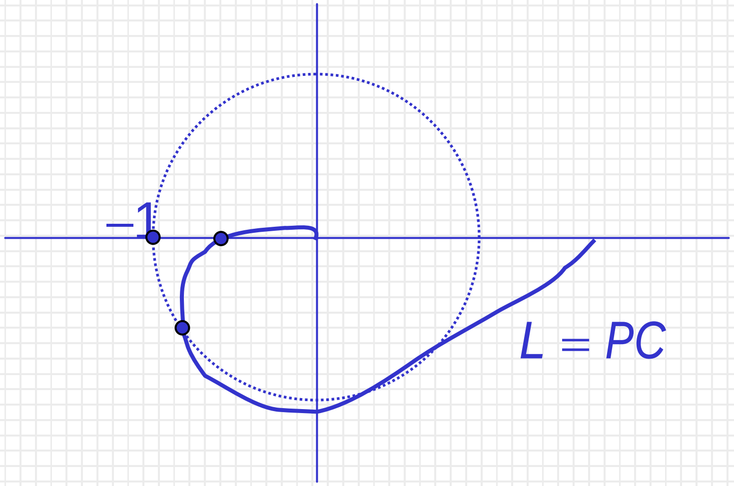
where

$$\mathcal{P} = \left\{ P(s) = \frac{k}{s(s+a)} : \quad k \in [1,10], \; a \in [1,10] \right\}.$$

The objective is to design a controller C such that it achieves

5.1. Margin Specs

In the complex plane, gain margin (*GM*) and phase margin (*PM*) have this graphical interpretation



We distinguish two types of margins: lower and upper. The lower type is shown above. The upper margins type is defined for conditionally stable systems (those with *PM* and *GM* that require minimum loop gain and phase; typical to unstable or multiple-integrator loops) . A system may require only lower margins or both.

5.1.1. Gain Margin

To convert *PM* and *GM* margins specs into closed-loop weights, let us represent the loop in a polar form

$$L(j\omega) = P(j\omega)C(j\omega) = \ell(\omega)e^{i\phi(\omega)}.$$

Now consider the complimentary sensitivity transfer function *T*

$$T = \frac{L}{1+L}$$

A lower *GM* is defined as

Plug the above into *T* and rearrange

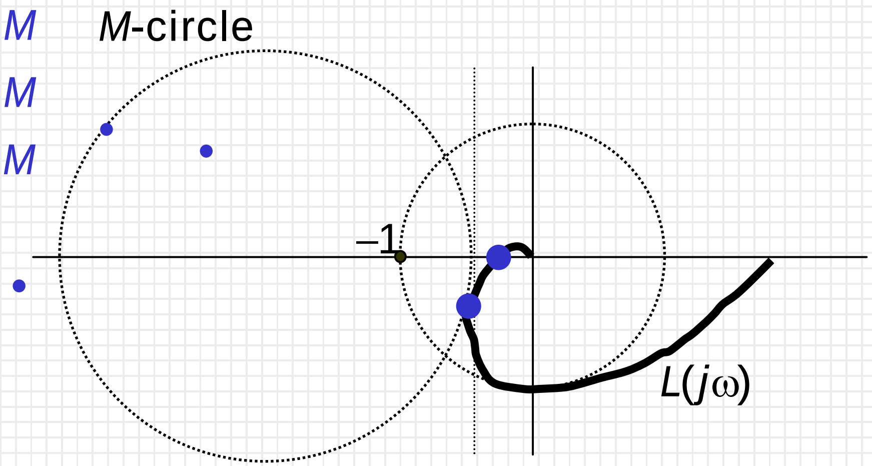
So

Note that

Hence, *GM* is bounded by

A graphical interpretation is shown below

$$\begin{cases} |T(j\omega)| > M \\ |T(j\omega)| = M \\ |T(j\omega)| < M \end{cases}$$



When does *GM* occur at max|T|?

In our example

Note that implying the larger the peaking in *T*, the smaller is *GM*.

5.1.2. Phase Margin

A lower *PM* is defined as

$$PM = 180^\circ + \angle L(j\omega_1) \quad (\angle L(j\omega_1) > -180^\circ), \qquad |L(j\omega_1)| = 1$$

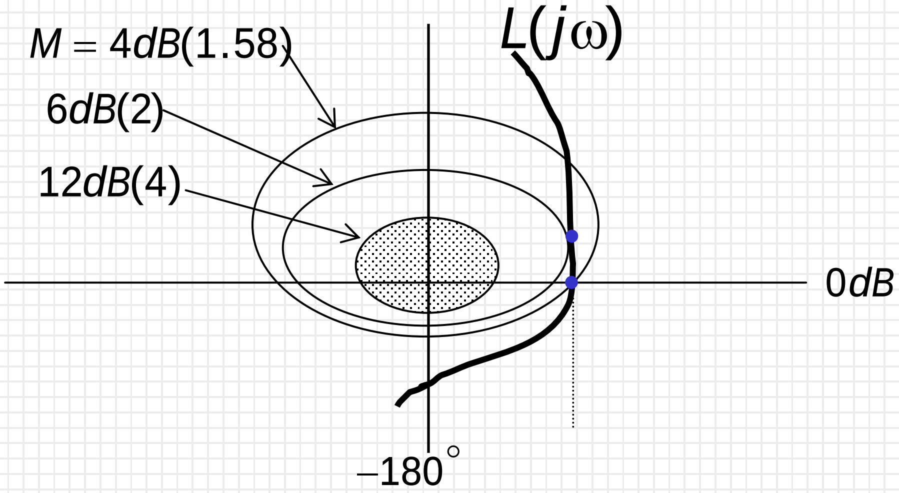
Again, plug into *T* and rearrange

$$|T(j\omega_1)| = \left|\frac{L(j\omega_1)}{1+L(j\omega_1)}\right| = \frac{1}{|1-\cos(\mathrm{PM})-i\sin(\mathrm{PM})|} = \frac{1}{\sqrt{(1-\cos(\mathrm{PM}))^2+(\sin(\mathrm{PM}))^2}}$$

Note that

so

Hence, *PM* is bounded by



Here

Note that implying the larger the peaking in *T*, the smaller is *PM*.

5.2. Manipulating Templates on NC

Before we can compute QFT bounds we must define plant templates. For instructive purposes, we first compute such bounda by manipulating a template on a NC.

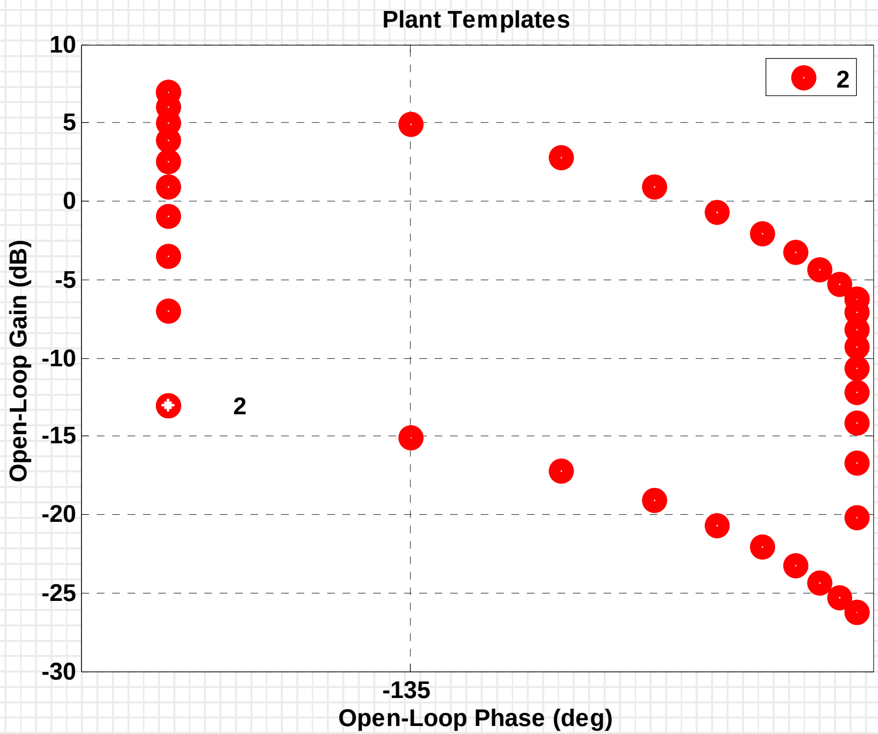
The general shape of the templates can often be computed analytically. In particular, it can be shown that in our example

only points in ∂Q map into $\partial \mathcal{P}(\omega)$.

To plot the template using the Toolbox:

1. Download `ch5_t.m`.

2. Run the file and enter 2 r/s for the frequency. Is this the template you have in hand (set axis manually to match those in the hard copy of NC)?



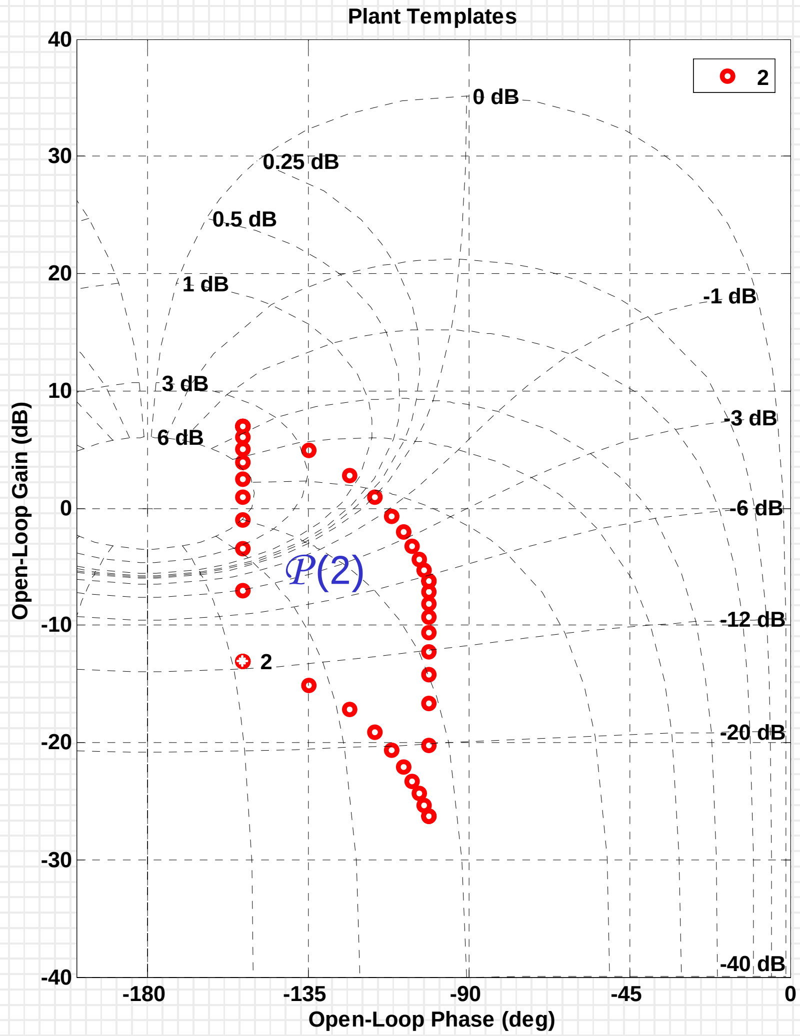
To plot a template with the correct scale relative to a working NC, do the following (the template figure being active):

- ngrid
- Edit|Axis Properties...: I used [-200,0,-40,40]
- File|Page Setup|Fill Page

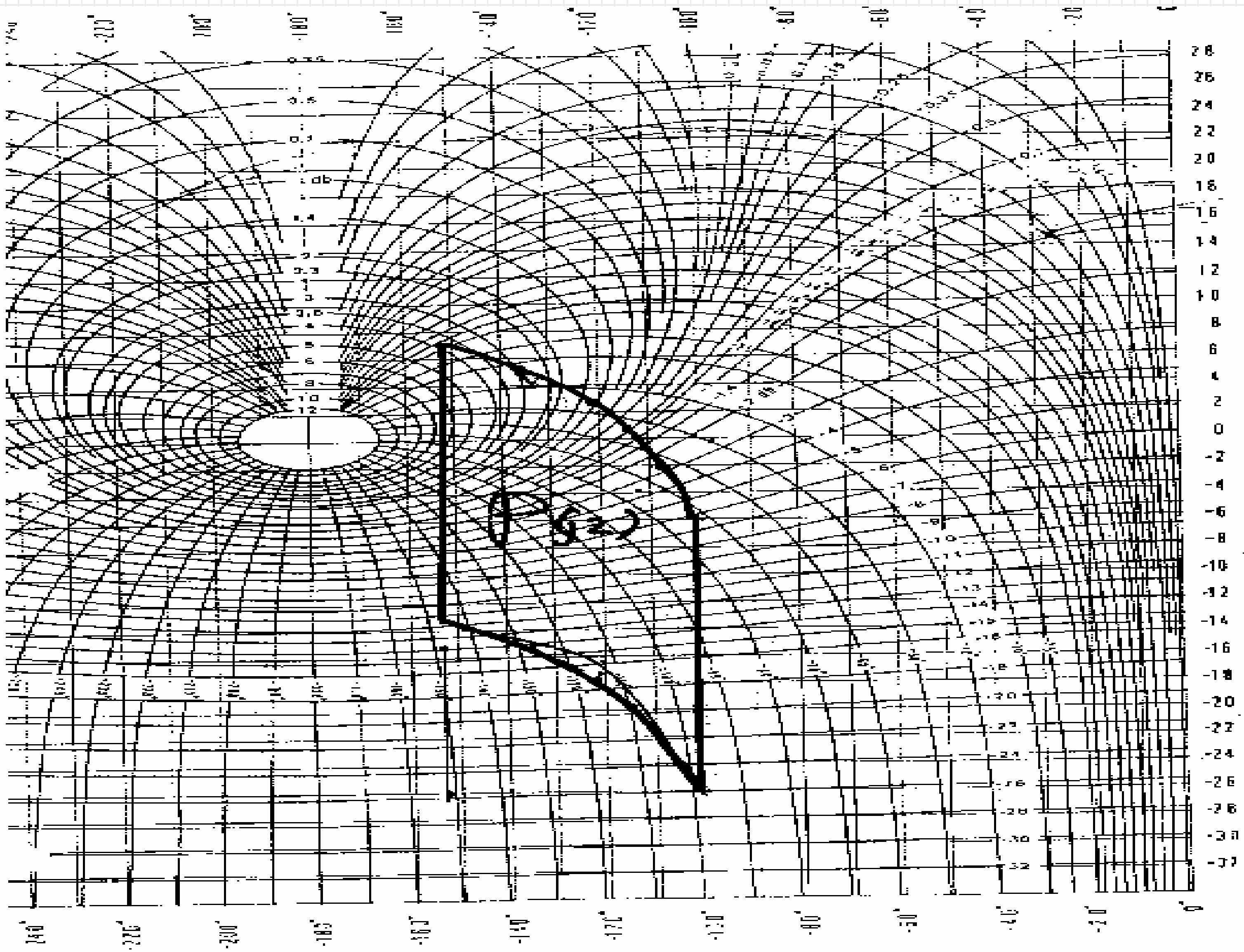
Note: ngrid generates a fixed grid. To modify, at the command line do:

```
>> type ngrid
```

and you will see how the grid is being generated.



Alternatively, we can manually plot the template on a given NC chart as done in the next page.



5.3. Manual Plotting of Bounds

Locate and mark the margins spec on your NC (M-circle).

Can we find a fixed controller $C(j2)$ such that the margin spec $|T(j2)| \le 1.2$ is satisfied for all possible plants? Yes. Here's how.

Note that since C is fixed ($L = CP$), loop variation satisfy

and on an NC

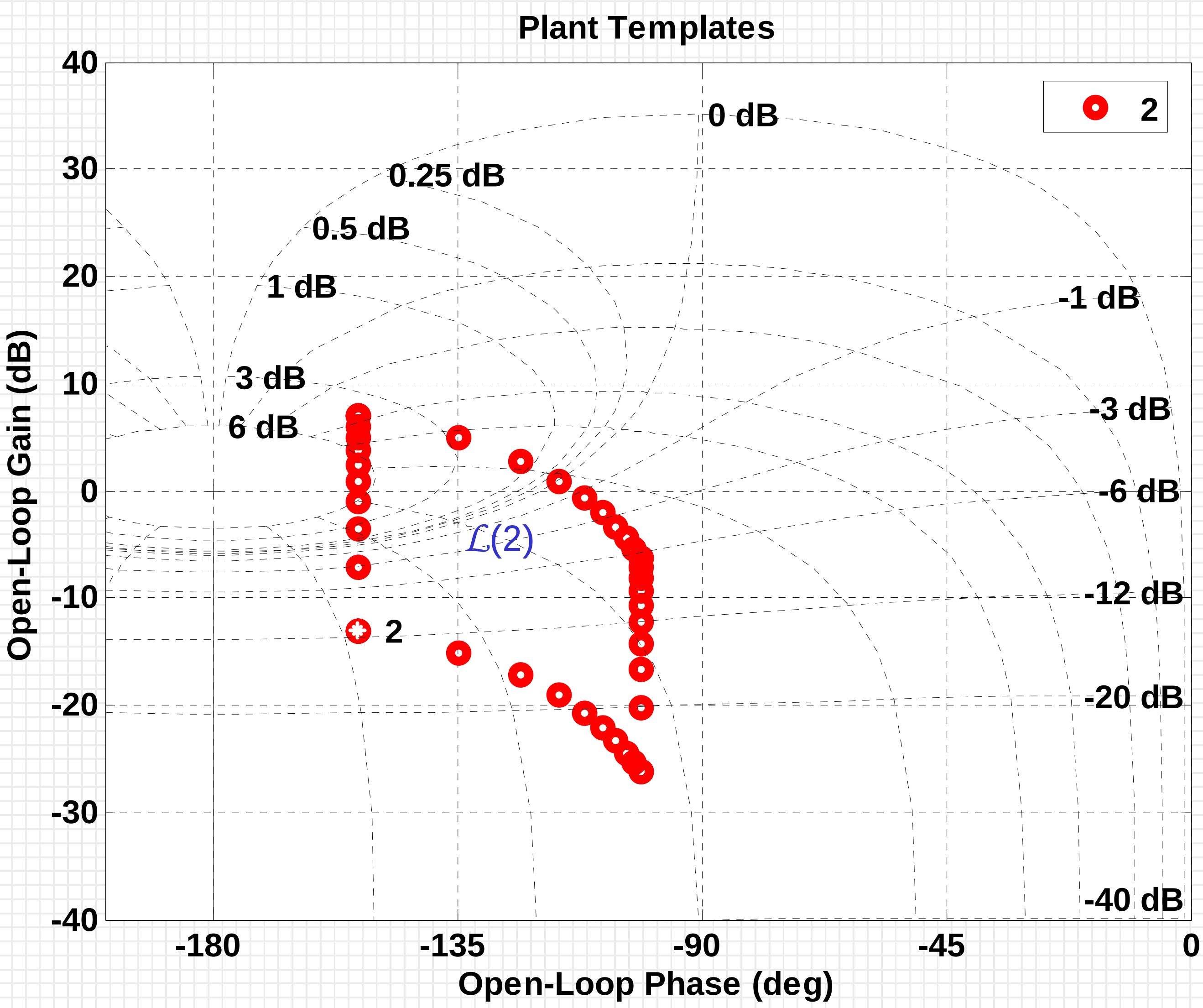
Hence, the loop template $\mathcal{L} = C\mathcal{P}$ has the same "shape" as that of $\mathcal{P}$, and moving $\mathcal{L}$ around an NC amounts to modifying the controller C. Each $P \in \mathcal{P}$ is shifted by a similar amount.

Next, place the template on the NC such that $\mathcal{L}$ lies away from the M-circle. What is the value of $C(j2)$ at this loop location? To find out, select a nominal plant from the plant family. For example, take

$$P_0 = \frac{k_0}{s(s+a_0)}, \quad a_0 = 1, k_0 = 1$$

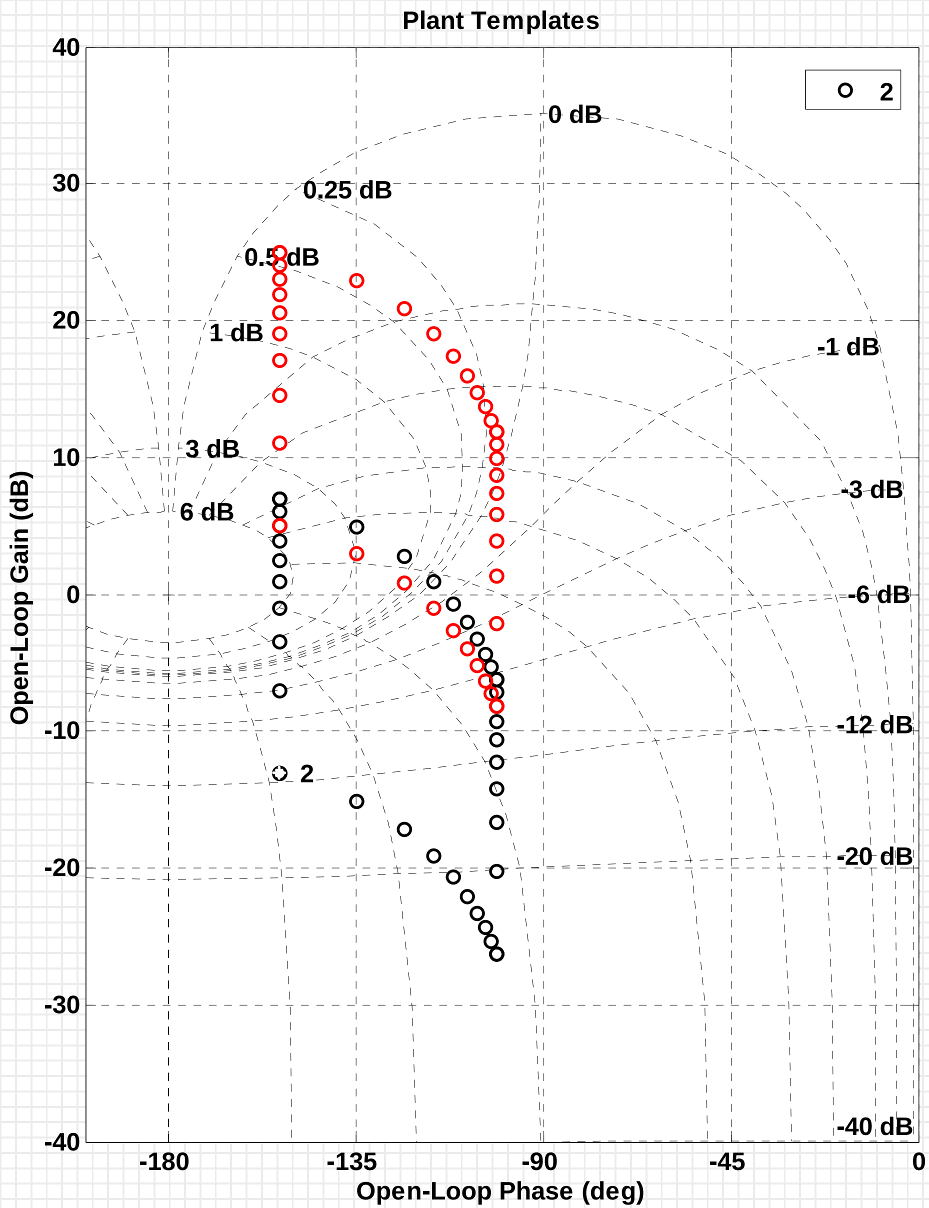
$$P_0(j2) = \text{-}0.2\text{-}i0.1 = \text{-}13\text{dB}\angle\text{-}154^\circ$$

which corresponds to case 1 in our plant LTI array (`nompt = 1`) or the lower point on the left edge. Say we located the loop such that $L_0(j2) = C(j2)\,P_0(j2) = \text{-}13\text{dB}\angle 154^\circ$, then (see next figure)


Plant Templates
Open-Loop Gain (dB)
Open-Loop Phase (deg)
40
30
20
10
0
-10
-20
-30
-40
-180
-135
-90
-45
0
0 dB
0.25 dB
0.5 dB
1 dB
3 dB
6 dB
-1 dB
-3 dB
-6 dB
-12 dB
-20 dB
-40 dB
2
$\mathcal{L}(2)$

Similarly, if $L_0(j2)$=3dB$\angle$-154°, then (see next figure)

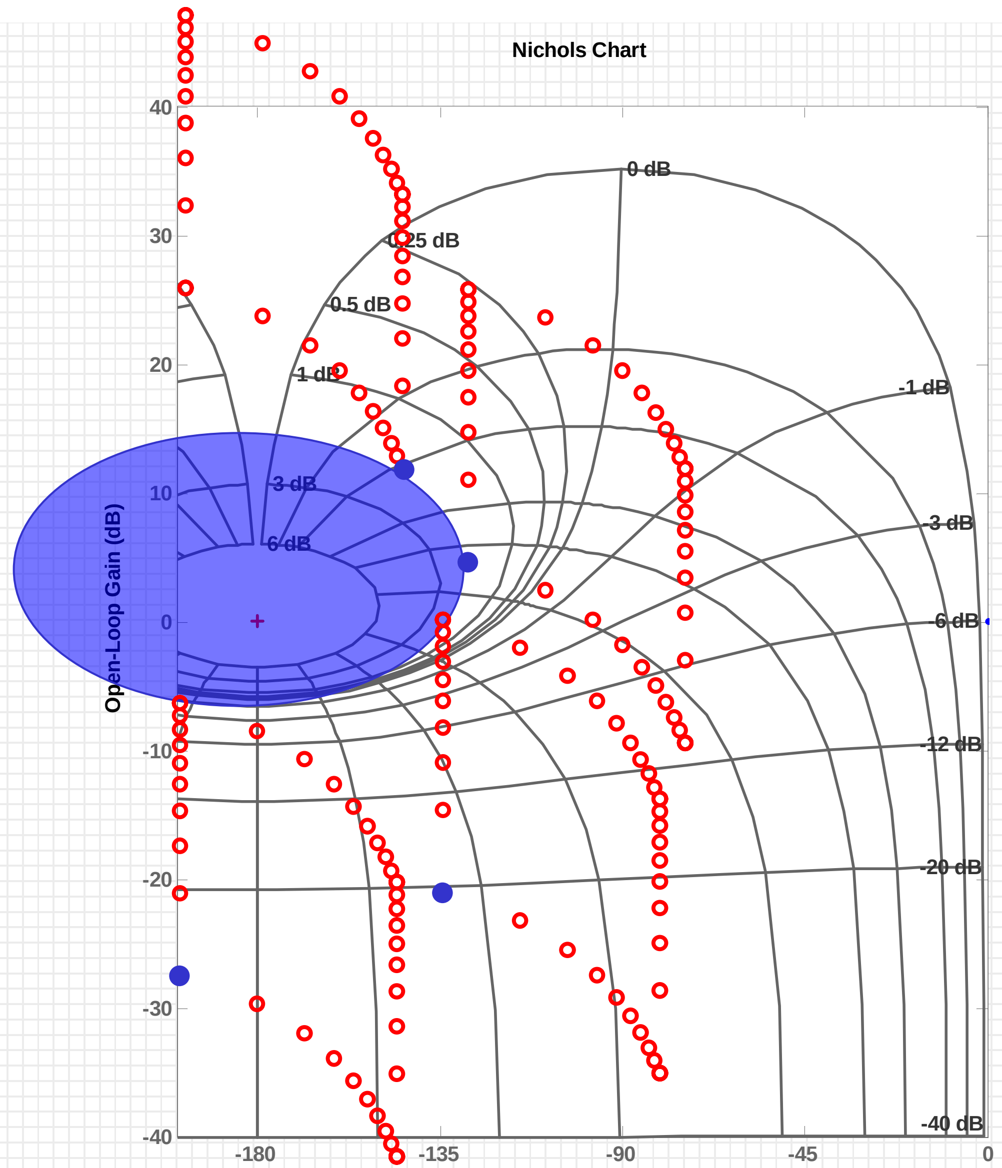
That is, vertical upward shift of L amounts to increasing the loop's gain which can happen only due an increase in the controller gain, |C |. Likewise, downward shift implies gain reduction. Horizontal shift to the right amount to increasing the controller phase, $\angle$C, and left shift decreases controller phase.

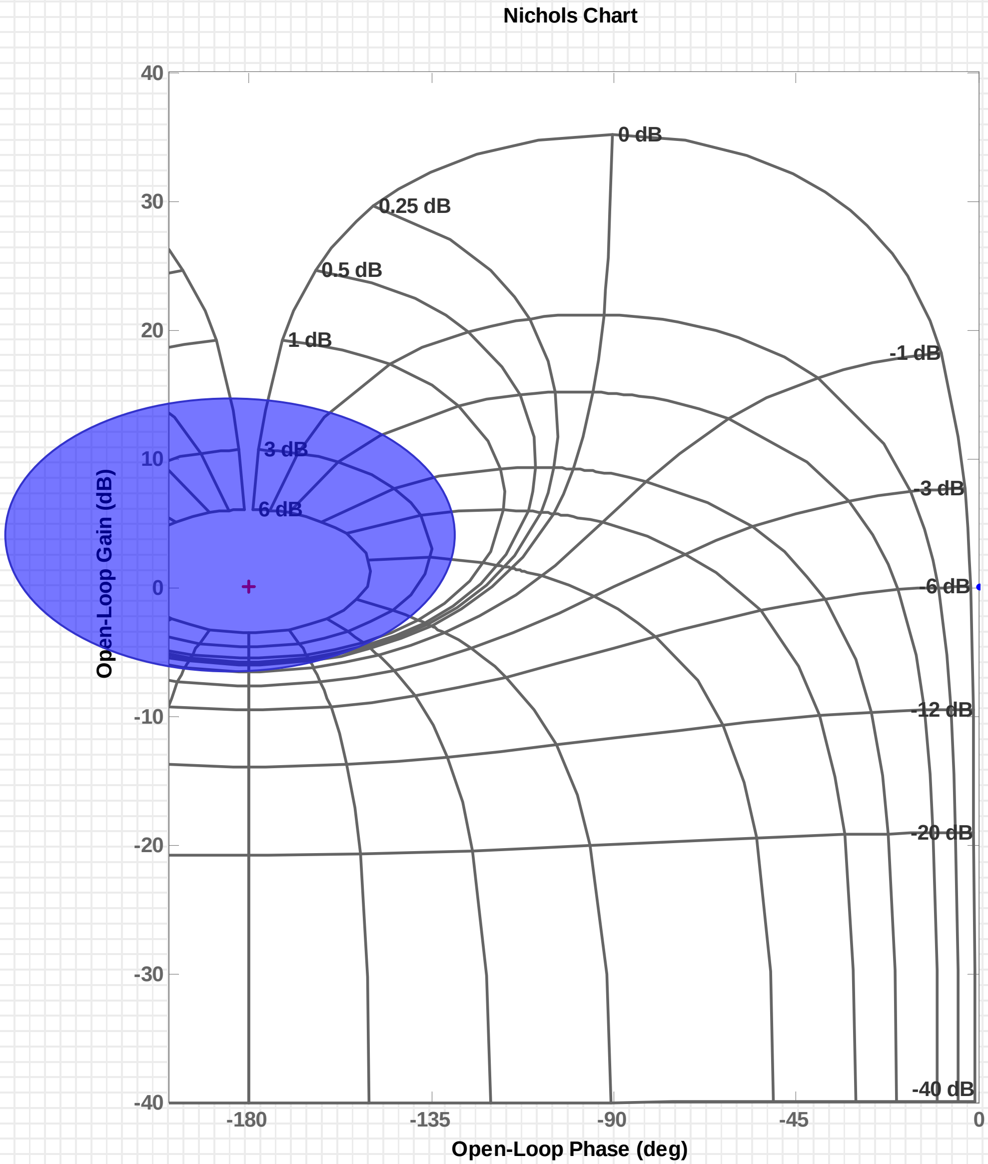

Plant Templates
2
0 dB
0.25 dB
0.5 dB
1 dB
3 dB
6 dB
-1 dB
-3 dB
-6 dB
-12 dB
-20 dB
-40 dB
2
40
30
20
10
0
-10
-20
-30
-40
Open-Loop Gain (dB)
-180
-135
-90
-45
0
Open-Loop Phase (deg)

Back to plotting the margin bound. Place $\mathcal{L}(j2)$ with $C = 1$ (i.e., $\mathcal{P}(j2)$) on the NC in the next page and clearly mark the M-circle . Our M-circle was 1.2 or 1.58dB. Interpolate if necessary.

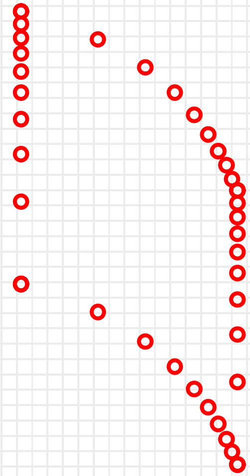
Now horizontally shift $\mathcal{L}(j2)$ to the left until its left edge touched the M-circle. Mark $L_0(j2)$ on the NC. Repeat by shifting $\mathcal{L}(j2)$ upwards while still touching the M-circle. Make sure to mark $L_0(j2)$ as you go along (every few dBs and degrees). Continue tracing the M-circle such that some point on $\partial\mathcal{L}(j2)$ always touches the M-circle. This may require both vertical and horizontal shifts. Continue to mark $L_0(j2)$ as you go along.

This is illustrated next.


Nichols Chart
Open-Loop Gain (dB)
Open-Loop Phase (deg)
40
30
20
10
0
-10
-20
-30
-40
-180
-135
-90
-45
0
0 dB
0.25 dB
0.5 dB
1 dB
3 dB
6 dB
-1 dB
-3 dB
-6 dB
-12 dB
-20 dB
-40 dB

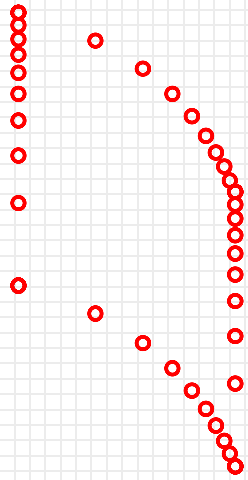


Nichols Chart
0 dB
0.25 dB
0.5 dB
1 dB
-1 dB
3 dB
6 dB
-3 dB
-6 dB
-12 dB
-20 dB
-40 dB
Open-Loop Gain (dB)
Open-Loop Phase (deg)



Print.

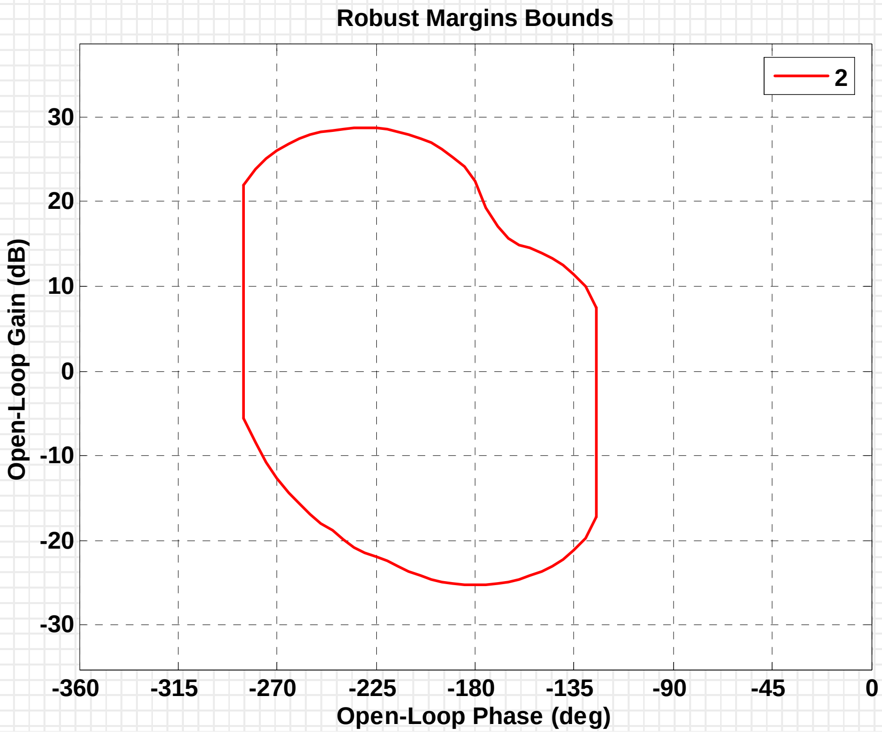
Print on a transparency film.

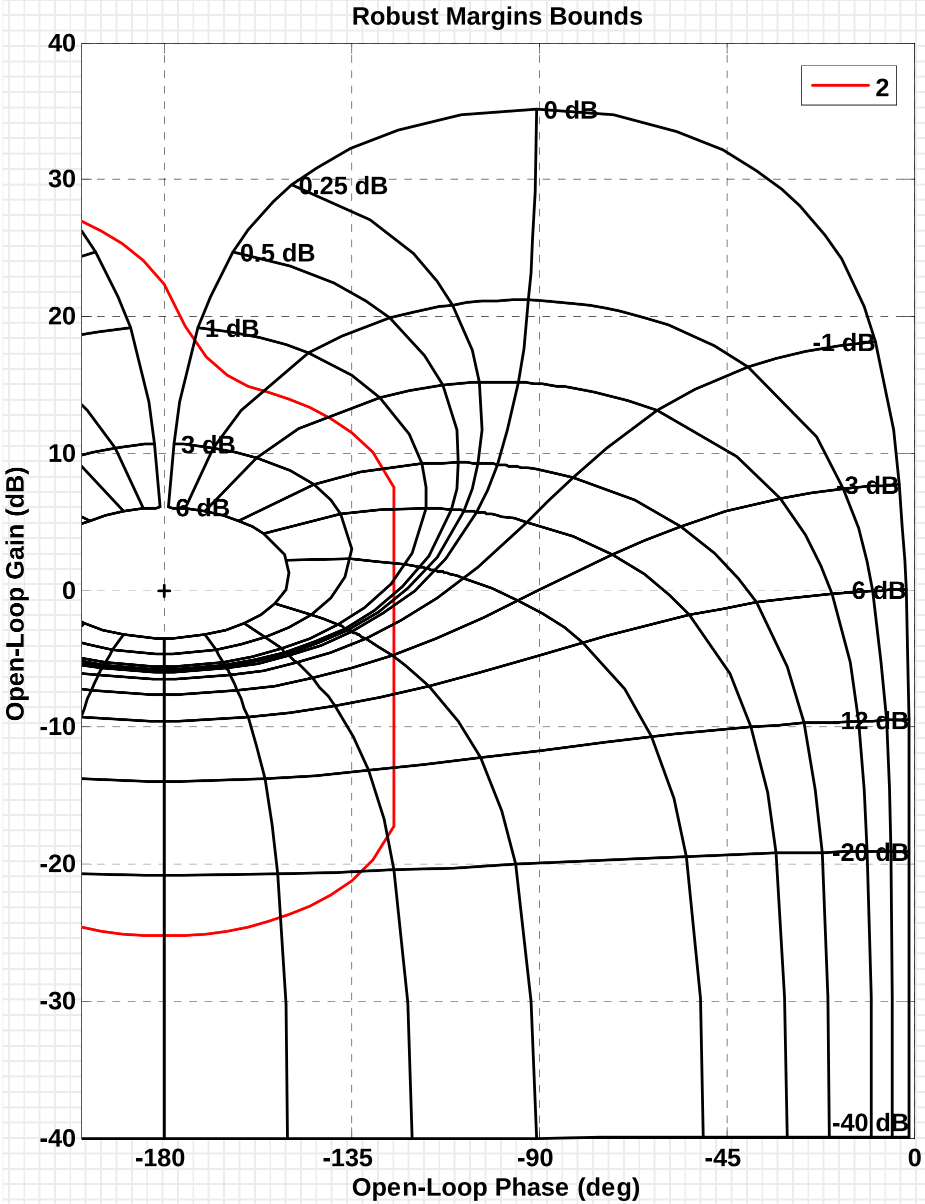


Get `ch5_b.m`. First create a template at w = 2 using `ch5_t.m`, the run this file which executes the following commands:

```
W1 = 1.2;
bdb1 = sisobnds(1,w,W1,P,[],nompt);
```

The result is shown to the right.

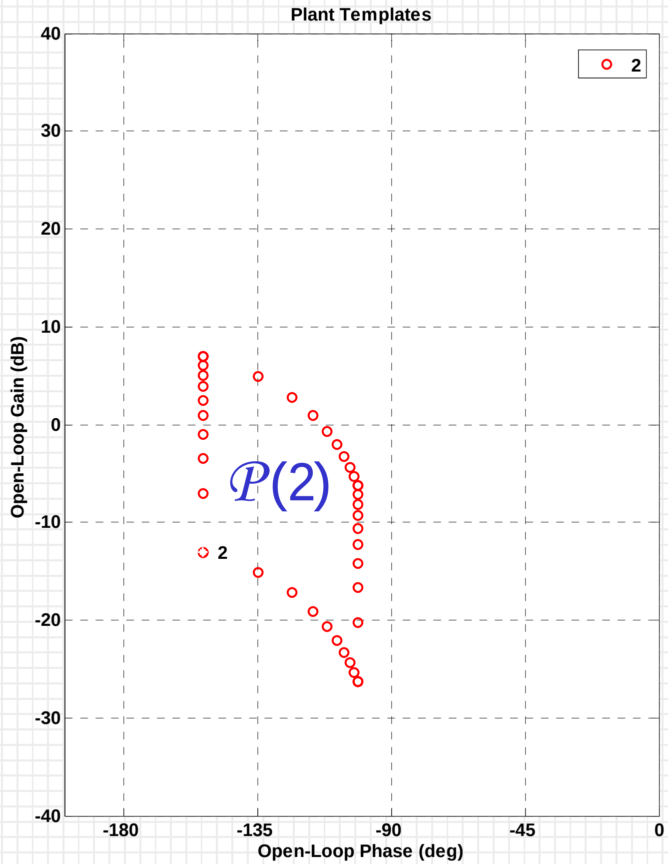




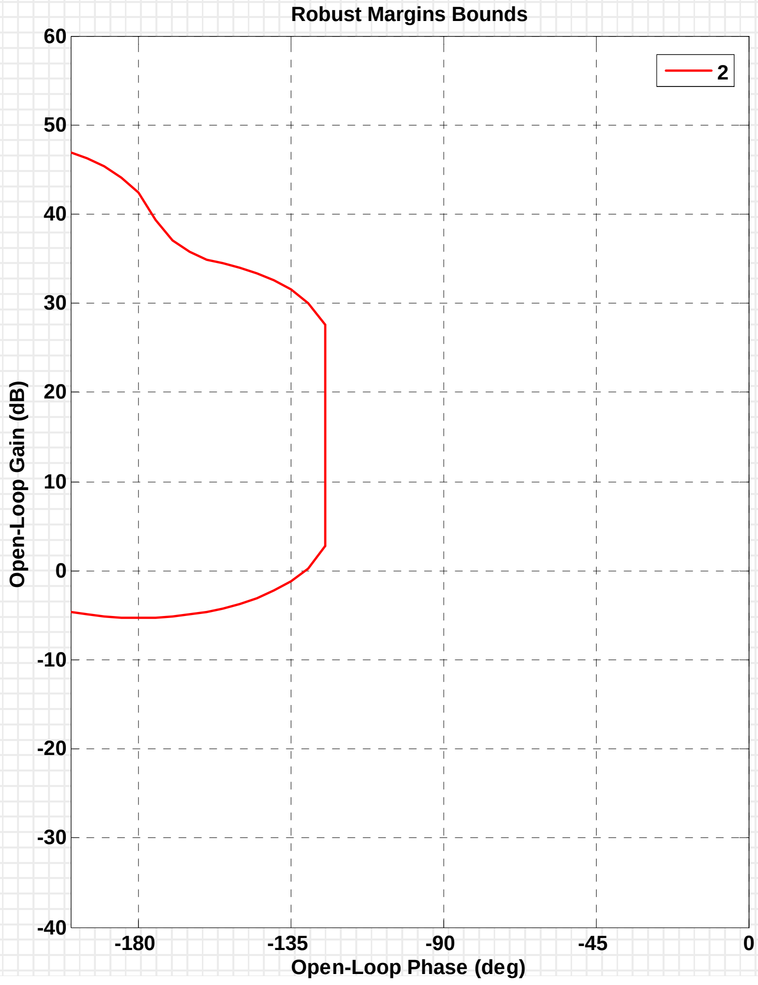
Now compare this analytically computed bound to your manually computed one. Modify figure properties as discussed earlier to obtain same scale.

What if our nominal plant is different? E.g.,

$$P_0 = \frac{k_0}{s(s+a_0)}, \quad a_0 = 1, k_0 = 10.$$



Simply determine the corresponding plant case (`nompt = 11`), and re-run `ch5_b.m`. This is what you should see.



Can you characterize the differences and similarities between the bounds computing using different nominal plants?